A Calculational Deductive System for Linear Temporal Logic

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This article surveys the linear temporal logic (LTL) literature and presents all the LTL theorems from the survey, plus many new ones, in a calculational deductive system. Calculational deductive systems, developed by Dijkstra and Scholten and extended by Gries and Schneider, are based on only four inference rules—Substitution, Leibniz, Equanimity, and Transitivity. Inference rules in the older Hilbert-style systems, notably modus ponens, appear as theorems in this calculational deductive system. This article extends the calculational deductive system of Gries and Schneider to LTL, using only the same four inference rules. Although space limitations preclude giving a proof of every theorem in this article, every theorem has been proved with calculational logic.

CCS Concepts: • **Theory of computation** → **Modal and temporal logics**;

Additional Key Words and Phrases: Calculational logic, equational logic, linear temporal logic

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Precedence Table

$[x := e]$ (textual substitution)	Highest precedence
$\neg \quad \bigcirc \quad \Diamond \quad \Box$	
$\mathcal{U} \quad \mathcal{W}$	
$=$ (conjunctional)	
$\vee \quad \wedge$	
$\Rightarrow \quad \Leftarrow$	
$\equiv$ (associative)	Lowest precedence

Definition of a model

A model σ is an infinite sequence of the form

$$\sigma : s_0, s_1, s_2, \ldots$$

where s_0 is the initial state and each state $s_i, 0 \leq i$ is the state at time i.

Example

σ	s_0	s_1	s_2	s_3	s_4	$\ldots$
x	8	9	10	11	12	$\ldots$
$x \geq 10$	F	F	T	T	T	$\ldots$

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The notation

$$(\sigma, j) \models p$$

means that the expression p holds at position j in a sequence σ.

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σ	s_0	s_1	s_2	s_3	s_4	$\ldots$
x	8	9	10	11	12	$\ldots$
$x \geq 10$	F	F	T	$\boxed{\text{T}}$	T	$\ldots$

$$(\sigma, 3) \models x \geq 10$$

The *next* operator $\bigcirc$

The semantics of the unary prefix operator $\bigcirc$ is

$$(\sigma, j) \models \bigcirc\, p \quad \text{iff} \quad (\sigma, j+1) \models p$$

That is, $\bigcirc\, p$ holds at position j iff p holds at position $j+1$.

σ	s_0	s_1	s_2	s_3	s_4	s_5	s_6	$\ldots$
x	8	9	10	11	12	13	14	$\ldots$
$10 \leq x < 13$	F	F	$\boxed{\text{T}}$	T	T	F	F	$\ldots$
$\bigcirc\, 10 \leq x < 13$	F	$\boxed{\text{T}}$	T	T	F	F	F	$\ldots$

$$(\sigma, 1) \models \bigcirc\, 10 \leq x < 13 \quad \text{because} \quad (\sigma, 2) \models 10 \leq x < 13$$

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σ	s_0	s_1	s_2	s_3	s_4	s_5	s_6	$\ldots$
x	8	9	10	11	12	13	14	$\ldots$
$10 \leq x < 13$	F	F	T	$\boxed{\text{T}}$	T	F	F	$\ldots$
$\bigcirc\ 10 \leq x < 13$	F	T	$\boxed{\text{T}}$	T	F	F	F	$\ldots$

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σ	s_0	s_1	s_2	s_3	s_4	s_5	s_6	$\ldots$
x	8	9	10	11	12	13	14	$\ldots$
$10 \leq x < 13$	F	F	T	T	$\boxed{\text{T}}$	F	F	$\ldots$
$\bigcirc\, 10 \leq x < 13$	F	T	T	$\boxed{\text{T}}$	F	F	F	$\ldots$

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σ	s_0	s_1	s_2	s_3	s_4	s_5	s_6	$\dots$
x	8	9	10	11	12	13	14	$\dots$
$10 \leq x < 13$	F	F	T	T	T	$\boxed{\mathrm{F}}$	F	$\dots$
$\bigcirc\, 10 \leq x < 13$	F	T	T	T	$\boxed{\mathrm{F}}$	F	F	$\dots$

The *until* operator $\mathcal{U}$

The semantics of the binary infix operator $\mathcal{U}$ is

$$(\sigma, j) \models p \,\mathcal{U}\, q \quad \text{iff}$$

$$(\exists k \mid k \geq j : (\sigma, k) \models q \wedge (\forall i \mid j \leq i < k : (\sigma, i) \models p))$$

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$$(\sigma, j) \models p \,\mathcal{U}\, q$$

σ	s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_8	$\ldots$
x	-1	0	1	2	3	4	5	6	7	8	$\ldots$
y	9	8	7	6	5	4	3	2	1	0	$\ldots$
$0 < x < y$											
$2 \leq y < 5$											
$(0 < x < y) \,\mathcal{U}\, (2 \leq y < 5)$											

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$$(\sigma, j) \models p\,\mathcal{U}\,q$$

σ	s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_8	$\ldots$
x	-1	0	1	2	3	4	5	6	7	8	$\ldots$
y	9	8	7	6	5	4	3	2	1	0	$\ldots$
$0 < x < y$	F	F	T	T	T	F	F	F	F	F	$\ldots$
$2 \leq y < 5$											
$(0 < x < y)\,\mathcal{U}\,(2 \leq y < 5)$											

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$$(\sigma, j) \models p\,\mathcal{U}\,q$$

σ	s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_8	$\ldots$
x	-1	0	1	2	3	4	5	6	7	8	$\ldots$
y	9	8	7	6	5	4	3	2	1	0	$\ldots$
$0 < x < y$	F	F	T	T	T	F	F	F	F	F	$\ldots$
$2 \leq y < 5$	F	F	F	F	F	T	T	T	F	F	$\ldots$
$(0 < x < y)\,\mathcal{U}\,(2 \leq y < 5)$											

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σ	s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_8	$\ldots$
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y	9	8	7	6	5	4	3	2	1	0	$\ldots$
$0 < x < y$	F	F	T	T	T	F	F	F	F	F	$\ldots$
$2 \leq y < 5$	F	F	F	F	F	T	T	T	F	F	$\ldots$
$(0 < x < y) \,\mathcal{U}\, (2 \leq y < 5)$	**?**										

$$(\exists k \mid k \geq j : (\sigma, k) \models q \wedge (\forall i \mid j \leq i < k : (\sigma, i) \models p))$$

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σ	s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_8	$\ldots$
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y	9	8	7	6	5	4	3	2	1	0	$\ldots$
$0 < x < y$	F	F	T	T	T	F	F	F	F	F	$\ldots$
$2 \leq y < 5$	F	F	F	F	F	T	T	T	F	F	$\ldots$
$(0 < x < y) \,\mathcal{U}\, (2 \leq y < 5)$	F										

$$(\exists k \mid k \geq j : (\sigma, k) \models q \wedge (\forall i \mid j \leq i < k : (\sigma, i) \models p))$$

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$$(\sigma, j) \models p \,\mathcal{U}\, q$$

σ	s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_8	$\ldots$
x	-1	0	1	2	3	4	5	6	7	8	$\ldots$
y	9	8	7	6	5	4	3	2	1	0	$\ldots$
$0 < x < y$	F	F	T	T	T	F	F	F	F	F	$\ldots$
$2 \leq y < 5$	F	F	F	F	F	T	T	T	F	F	$\ldots$
$(0 < x < y)\,\mathcal{U}\,(2 \leq y < 5)$	F	F									

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y	9	8	7	6	5	4	3	2	1	0	$\ldots$
$0 < x < y$	F	F	T	T	T	F	F	F	F	F	$\ldots$
$2 \leq y < 5$	F	F	F	F	F	T	T	T	F	F	$\ldots$
$(0 < x < y) \,\mathcal{U}\, (2 \leq y < 5)$	F	F	T								

$$(\exists k \mid k \geq j : (\sigma, k) \models q \wedge (\forall i \mid j \leq i < k : (\sigma, i) \models p))$$

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σ	s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_8	$\ldots$
x	-1	0	1	2	3	4	5	6	7	8	$\ldots$
y	9	8	7	6	5	4	3	2	1	0	$\ldots$
$0 < x < y$	F	F	T	T	T	F	F	F	F	F	$\ldots$
$2 \leq y < 5$	F	F	F	F	F	T	T	T	F	F	$\ldots$
$(0 < x < y) \,\mathcal{U}\, (2 \leq y < 5)$	F	F	T	T							

$$(\exists k \mid k \geq j : (\sigma, k) \models q \land (\forall i \mid j \leq i < k : (\sigma, i) \models p))$$

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x	-1	0	1	2	3	4	5	6	7	8	$\ldots$
y	9	8	7	6	5	4	3	2	1	0	$\ldots$
$0 < x < y$	F	F	T	T	$\boxed{\mathrm{T}}$	F	F	F	F	F	$\ldots$
$2 \leq y < 5$	F	F	F	F	F	$\boxed{\mathrm{T}}$	T	T	F	F	$\ldots$
$(0 < x < y) \,\mathcal{U}\, (2 \leq y < 5)$	F	F	T	T	T						

$$(\exists k \mid k \geq j : (\sigma, k) \models q \wedge (\forall i \mid j \leq i < k : (\sigma, i) \models p))$$

$$(\sigma, j) \models p \,\mathcal{U}\, q$$

σ	s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_8	$\ldots$
x	-1	0	1	2	3	4	5	6	7	8	$\ldots$
y	9	8	7	6	5	4	3	2	1	0	$\ldots$
$0 < x < y$	F	F	T	T	T	F	F	F	F	F	$\ldots$
$2 \leq y < 5$	F	F	F	F	F	$\boxed{\mathrm{T}}$	T	T	F	F	$\ldots$
$(0 < x < y)\,\mathcal{U}\,(2 \leq y < 5)$	F	F	T	T	T	**?**					

$$(\exists k \mid k \geq j : (\sigma, k) \models q \wedge (\forall i \mid j \leq i < k : (\sigma, i) \models p))$$

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$$(\sigma, j) \models p \,\mathcal{U}\, q$$

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x	-1	0	1	2	3	4	5	6	7	8	$\ldots$
y	9	8	7	6	5	4	3	2	1	0	$\ldots$
$0 < x < y$	F	F	T	T	T	F	F	F	F	F	$\ldots$
$2 \leq y < 5$	F	F	F	F	F	$\boxed{\mathrm{T}}$	T	T	F	F	$\ldots$
$(0 < x < y)\,\mathcal{U}\,(2 \leq y < 5)$	F	F	T	T	T	**?**					

What is p*U*q when k = j, q ≡ true, and p ≡ false?

$$(\exists k \mid k \geq j : (\sigma, k) \models q \wedge (\forall i \mid j \leq i < k : (\sigma, i) \models p))$$

A Calculational Deductive System for Linear Temporal Logic

$$(\sigma, j) \models p\,\mathcal{U}\,q$$

σ	s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_8	$\dots$
x	-1	0	1	2	3	4	5	6	7	8	$\dots$
y	9	8	7	6	5	4	3	2	1	0	$\dots$
$0 < x < y$	F	F	T	T	T	F	F	F	F	F	$\dots$
$2 \leq y < 5$	F	F	F	F	F	$\boxed{\mathrm{T}}$	T	T	F	F	$\dots$
$(0 < x < y)\,\mathcal{U}\,(2 \leq y < 5)$	F	F	T	T	T	**?**					

What is p*U*q when k = j, q ≡ true, and p ≡ false?

$$(\exists k \mid k \geq j : \underbrace{(\sigma, k) \models q}_{\text{true}} \wedge (\forall i \mid j \leq i < k : (\sigma, i) \models p))$$

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$$(\sigma, j) \models p \, \mathcal{U} \, q$$

σ	s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_8	$\dots$
x	-1	0	1	2	3	4	5	6	7	8	$\dots$
y	9	8	7	6	5	4	3	2	1	0	$\dots$
$0 < x < y$	F	F	T	T	T	F	F	F	F	F	$\dots$
$2 \leq y < 5$	F	F	F	F	F	$\boxed{\text{T}}$	T	T	F	F	$\dots$
$(0 < x < y) \, \mathcal{U} \, (2 \leq y < 5)$	F	F	T	T	T	?					

What is p*U*q when k = j, q ≡ true, and p ≡ false?

$$(\exists k \mid k \geq j : \underbrace{(\sigma, k) \models q}_{\text{true}} \wedge (\forall i \mid j \leq i < k : \underbrace{(\sigma, i) \models p}_{\text{false}}))$$

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$$(\sigma, j) \models p \,\mathcal{U}\, q$$

σ	s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_8	$\ldots$
x	-1	0	1	2	3	4	5	6	7	8	$\ldots$
y	9	8	7	6	5	4	3	2	1	0	$\ldots$
$0 < x < y$	F	F	T	T	T	F	F	F	F	F	$\ldots$
$2 \leq y < 5$	F	F	F	F	F	$\boxed{\mathrm{T}}$	T	T	F	F	$\ldots$
$(0 < x < y)\,\mathcal{U}\,(2 \leq y < 5)$	F	F	T	T	T	?					

What is p*U*q when k = j, q ≡ true, and p ≡ false?

$$(\exists k \mid k \geq j : \underbrace{(\sigma, k) \models q}_{\text{true}} \wedge (\forall i \mid \underbrace{j \leq i < k}_{\text{false}} : \underbrace{(\sigma, i) \models p}_{\text{false}}))$$

$$(\sigma, j) \models p \,\mathcal{U}\, q$$

σ	s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_8	$\ldots$
x	-1	0	1	2	3	4	5	6	7	8	$\ldots$
y	9	8	7	6	5	4	3	2	1	0	$\ldots$
$0 < x < y$	F	F	T	T	T	F	F	F	F	F	$\ldots$
$2 \leq y < 5$	F	F	F	F	F	$\boxed{\mathrm{T}}$	T	T	F	F	$\ldots$
$(0 < x < y)\,\mathcal{U}\,(2 \leq y < 5)$	F	F	T	T	T	T					

The “empty range rule”

$$(\exists k \mid k \geq j : (\sigma, k) \models q \wedge (\forall i \mid \underbrace{j \leq i < k}_{\text{false}} : (\sigma, i) \models p))$$

$$(\sigma, j) \models p \,\mathcal{U}\, q$$

σ	s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_8	$\ldots$
x	-1	0	1	2	3	4	5	6	7	8	$\ldots$
y	9	8	7	6	5	4	3	2	1	0	$\ldots$
$0 < x < y$	F	F	T	T	T	F	F	F	F	F	$\ldots$
$2 \leq y < 5$	F	F	F	F	F	T	$\boxed{\text{T}}$	T	F	F	$\ldots$
$(0 < x < y) \,\mathcal{U}\, (2 \leq y < 5)$	F	F	T	T	T	T	T				

$$(\exists k \mid k \geq j : (\sigma, k) \models q \wedge (\forall i \mid j \leq i < k : (\sigma, i) \models p))$$

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$$(\sigma, j) \models p \,\mathcal{U}\, q$$

σ	s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_8	$\ldots$
x	-1	0	1	2	3	4	5	6	7	8	$\ldots$
y	9	8	7	6	5	4	3	2	1	0	$\ldots$
$0 < x < y$	F	F	T	T	T	F	F	F	F	F	$\ldots$
$2 \leq y < 5$	F	F	F	F	F	T	T	$\boxed{\mathrm{T}}$	F	F	$\ldots$
$(0 < x < y)\,\mathcal{U}\,(2 \leq y < 5)$	F	F	T	T	T	T	T	T			

$$(\exists k \mid k \geq j : (\sigma, k) \models q \wedge (\forall i \mid j \leq i < k : (\sigma, i) \models p))$$

A Calculational Deductive System for Linear Temporal Logic

$$(\sigma, j) \models p \, \mathcal{U} \, q$$

σ	s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_8	$\ldots$
x	-1	0	1	2	3	4	5	6	7	8	$\ldots$
y	9	8	7	6	5	4	3	2	1	0	$\ldots$
$0 < x < y$	F	F	T	T	T	F	F	F	F	F	$\ldots$
$2 \leq y < 5$	F	F	F	F	F	T	T	T	F	F	$\ldots$
$(0 < x < y) \, \mathcal{U} \, (2 \leq y < 5)$	F	F	T	T	T	T	T	T	F		

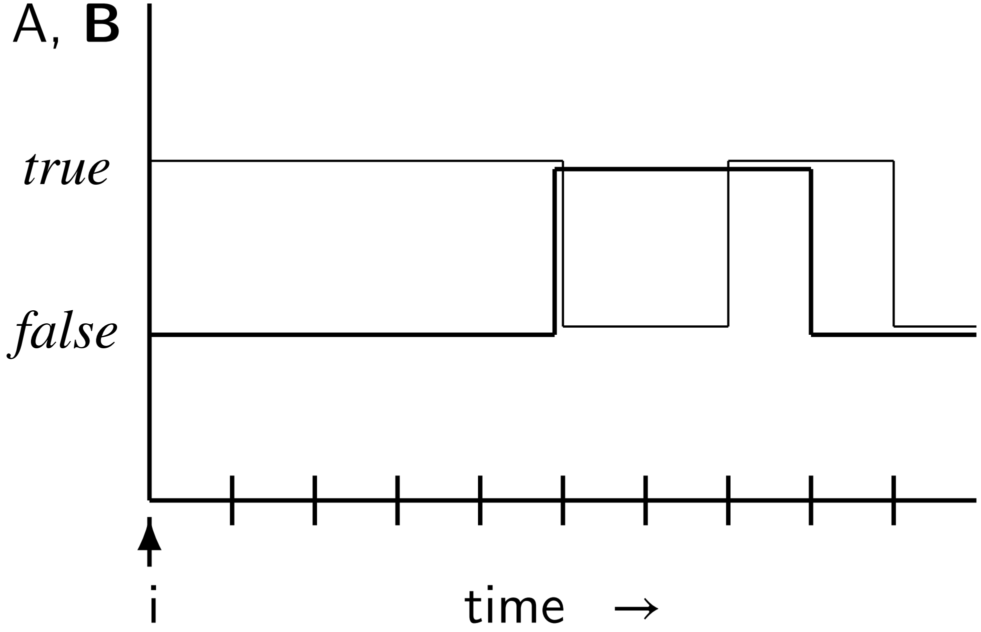
$$(\exists k \mid k \geq j : (\sigma, k) \models q \wedge (\forall i \mid j \leq i < k : (\sigma, i) \models p))$$

$$(\sigma, j) \models p \,\mathcal{U}\, q$$

σ	s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_8	$\ldots$
x	-1	0	1	2	3	4	5	6	7	8	$\ldots$
y	9	8	7	6	5	4	3	2	1	0	$\ldots$
$0 < x < y$	F	F	T	T	T	F	F	F	F	$\boxed{\text{F}}$	$\ldots$
$2 \leq y < 5$	F	F	F	F	F	T	T	T	F	F	$\ldots$
$(0 < x < y)\,\mathcal{U}\,(2 \leq y < 5)$	F	F	T	T	T	T	T	T	F	F	$\ldots$

$$(\exists k \mid k \geq j : (\sigma, k) \models q \wedge (\forall i \mid j \leq i < k : (\sigma, i) \models p))$$

$A \mathcal{U} B$



The *eventually* operator $\Diamond$

The semantics of the unary prefix operator $\Diamond$ is

$$(\sigma, j) \models \Diamond p \quad \text{iff} \quad (\exists k \mid k \geq j : (\sigma, k) \models p)$$

σ	s_0	s_1	s_2	s_3	s_4	s_5	s_6	$\ldots$
x	1	2	3	4	5	6	7	$\ldots$
$3 \leq x < 6$	F	F	T	T	T	F	F	$\ldots$
$\Diamond (3 \leq x < 6)$								

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σ	s_0	s_1	s_2	s_3	s_4	s_5	s_6	$\ldots$
x	1	2	3	4	5	6	7	$\ldots$
$3 \leq x < 6$	F	F	$\boxed{\text{T}}$	$\boxed{\text{T}}$	$\boxed{\text{T}}$	F	F	$\ldots$
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σ	s_0	s_1	s_2	s_3	s_4	s_5	s_6	$\ldots$
x	1	2	3	4	5	6	7	$\ldots$
$3 \leq x < 6$	F	F	$\boxed{\text{T}}$	$\boxed{\text{T}}$	$\boxed{\text{T}}$	F	F	$\ldots$
$\Diamond\,(3 \leq x < 6)$	T	T						

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x	1	2	3	4	5	6	7	$\ldots$
$3 \leq x < 6$	F	F	$\boxed{\text{T}}$	$\boxed{\text{T}}$	$\boxed{\text{T}}$	F	F	$\ldots$
$\Diamond(3 \leq x < 6)$	T	T	T					

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x	1	2	3	4	5	6	7	$\ldots$
$3 \leq x < 6$	F	F	T	$\boxed{\text{T}}$	$\boxed{\text{T}}$	F	F	$\ldots$
$\Diamond\,(3 \leq x < 6)$	T	T	T	T				

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x	1	2	3	4	5	6	7	$\ldots$
$3 \leq x < 6$	F	F	T	T	$\boxed{\text{T}}$	F	F	$\ldots$
$\Diamond(3 \leq x < 6)$	T	T	T	T	T			

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The semantics of the unary prefix operator $\Diamond$ is

$$(\sigma, j) \models \Diamond p \quad \text{iff} \quad (\exists k \mid k \geq j : (\sigma, k) \models p)$$

σ	s_0	s_1	s_2	s_3	s_4	s_5	s_6	$\ldots$
x	1	2	3	4	5	6	7	$\ldots$
$3 \leq x < 6$	F	F	T	T	T	F	F	$\ldots$
$\Diamond(3 \leq x < 6)$	T	T	T	T	T	F		

The *eventually* operator $\Diamond$

The semantics of the unary prefix operator $\Diamond$ is

$$(\sigma, j) \models \Diamond\, p \quad \text{iff} \quad (\exists k \mid k \geq j : (\sigma, k) \models p)$$

σ	s_0	s_1	s_2	s_3	s_4	s_5	s_6	$\ldots$
x	1	2	3	4	5	6	7	$\ldots$
$3 \leq x < 6$	F	F	T	T	T	F	F	$\ldots$
$\Diamond\,(3 \leq x < 6)$	T	T	T	T	T	F	F	$\ldots$

A Calculational Deductive System for Linear Temporal Logic

σ	s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	$\ldots$
p	F	F	T	F	F	T	F	F	F	F	$\ldots$
q	F	F	T	T	F	F	T	T	F	F	$\ldots$
$\Diamond p$											
$\Diamond q$											

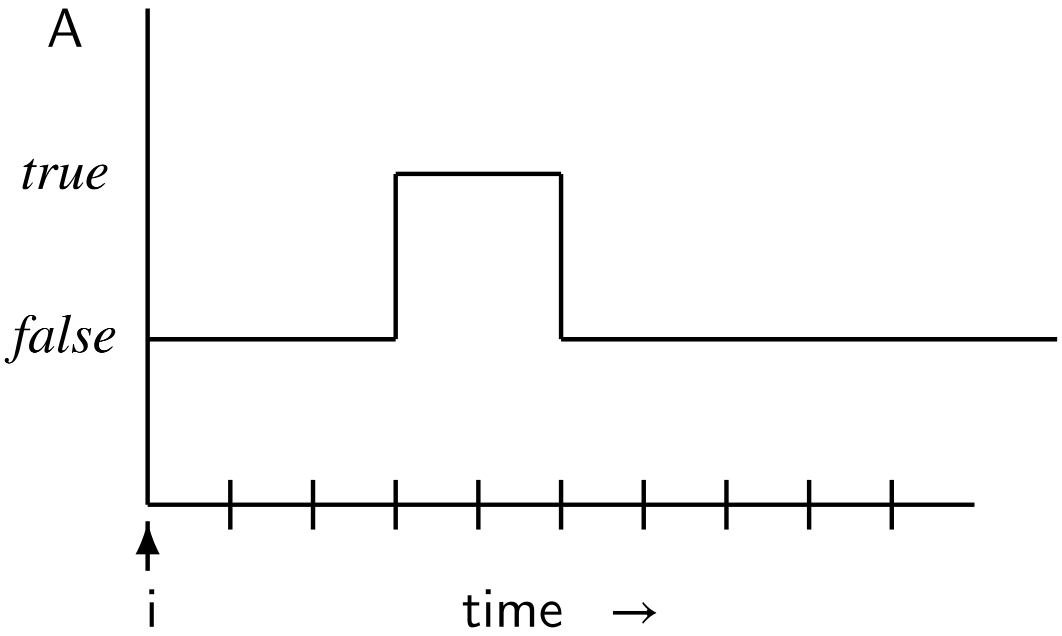
A Calculational Deductive System for Linear Temporal Logic

σ	s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	$\ldots$
p	F	F	T	F	F	T	F	F	F	F	$\ldots$
q	F	F	T	T	F	F	T	T	F	F	$\ldots$
$\Diamond p$	T	T	T	T	T	T	F	F	F	F	$\ldots$
$\Diamond q$											

A Calculational Deductive System for Linear Temporal Logic

σ	s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	$\ldots$
p	F	F	T	F	F	T	F	F	F	F	$\ldots$
q	F	F	T	T	F	F	T	T	F	F	$\ldots$
$\Diamond p$	T	T	T	T	T	T	F	F	F	F	$\ldots$
$\Diamond q$	T	T	T	T	T	T	T	T	T	T	$\ldots$

$\Diamond A$



$\Diamond A$ is a liveness property.

Example: $p2 \Rightarrow \Diamond p4$

Algorithm 4.1: Third attempt

boolean wantp ← false, wantq ← false

p

```
    loop forever
p1:     non-critical section
p2:     wantp ← true
p3:     await wantq = false
p4:     critical section
p5:     wantp ← false
```

q

```
    loop forever
q1:     non-critical section
q2:     wantq ← true
q3:     await wantp = false
q4:     critical section
q5:     wantq ← false
```

A Calculational Deductive System for Linear Temporal Logic

The *always* operator $\Box$

The semantics of the unary prefix operator $\Box$ is

$$(\sigma, j) \models \Box\, p \quad \text{iff} \quad (\forall k \mid k \geq j : (\sigma, k) \models p)$$

σ	s_0	s_1	s_2	s_3	s_4	s_5	s_6	$\ldots$
x	1	2	3	4	5	6	7	$\ldots$
$x \geq 4$	F	F	F	T	T	T	T	$\ldots$
$\Box\,(x \geq 4)$								

The *always* operator $\Box$

The semantics of the unary prefix operator $\Box$ is

$$(\sigma, j) \models \Box\, p \quad \text{iff} \quad (\forall k \mid k \geq j : (\sigma, k) \models p)$$

σ	s_0	s_1	s_2	s_3	s_4	s_5	s_6	$\dots$
x	1	2	3	4	5	6	7	$\dots$
$x \geq 4$	F	F	F	T	T	T	T	$\dots$
$\Box\,(x \geq 4)$	F							

A Calculational Deductive System for Linear Temporal Logic

The *always* operator $\Box$

The semantics of the unary prefix operator $\Box$ is

$$(\sigma, j) \models \Box\, p \quad \text{iff} \quad (\forall k \mid k \geq j : (\sigma, k) \models p)$$

σ	s_0	s_1	s_2	s_3	s_4	s_5	s_6	$\ldots$
x	1	2	3	4	5	6	7	$\ldots$
$x \geq 4$	F	F	F	T	T	T	T	$\ldots$
$\Box\,(x \geq 4)$	F	F						

A Calculational Deductive System for Linear Temporal Logic

The *always* operator $\Box$

The semantics of the unary prefix operator $\Box$ is

$$(\sigma, j) \models \Box\, p \quad \text{iff} \quad (\forall k \mid k \geq j : (\sigma, k) \models p)$$

σ	s_0	s_1	s_2	s_3	s_4	s_5	s_6	$\ldots$
x	1	2	3	4	5	6	7	$\ldots$
$x \geq 4$	F	F	F	T	T	T	T	$\ldots$
$\Box\,(x \geq 4)$	F	F	F					

A Calculational Deductive System for Linear Temporal Logic

The *always* operator $\Box$

The semantics of the unary prefix operator $\Box$ is

$$(\sigma, j) \models \Box\, p \quad \text{iff} \quad (\forall k \mid k \geq j : (\sigma, k) \models p)$$

σ	s_0	s_1	s_2	s_3	s_4	s_5	s_6	$\ldots$
x	1	2	3	4	5	6	7	$\ldots$
$x \geq 4$	F	F	F	T	T	T	T	$\ldots$
$\Box\,(x \geq 4)$	F	F	F	T				

A Calculational Deductive System for Linear Temporal Logic

The *always* operator $\Box$

The semantics of the unary prefix operator $\Box$ is

$$(\sigma, j) \models \Box\, p \quad \text{iff} \quad (\forall k \mid k \geq j : (\sigma, k) \models p)$$

σ	s_0	s_1	s_2	s_3	s_4	s_5	s_6	$\ldots$
x	1	2	3	4	5	6	7	$\ldots$
$x \geq 4$	F	F	F	T	T	T	T	$\ldots$
$\Box\,(x \geq 4)$	F	F	F	T	T			

The *always* operator □

The semantics of the unary prefix operator □ is

$$(\sigma, j) \models \Box\, p \quad \text{iff} \quad (\forall k \mid k \geq j : (\sigma, k) \models p)$$

σ	s_0	s_1	s_2	s_3	s_4	s_5	s_6	$\ldots$
x	1	2	3	4	5	6	7	$\ldots$
$x \geq 4$	F	F	F	T	T	T	T	$\ldots$
$\Box\,(x \geq 4)$	F	F	F	T	T	T		

A Calculational Deductive System for Linear Temporal Logic

The *always* operator $\Box$

The semantics of the unary prefix operator $\Box$ is

$$(\sigma, j) \models \Box\, p \quad \text{iff} \quad (\forall k \mid k \geq j : (\sigma, k) \models p)$$

σ	s_0	s_1	s_2	s_3	s_4	s_5	s_6	$\dots$
x	1	2	3	4	5	6	7	$\dots$
$x \geq 4$	F	F	F	T	T	T	T	$\dots$
$\Box\,(x \geq 4)$	F	F	F	T	T	T	T	

A Calculational Deductive System for Linear Temporal Logic

σ	s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	$\ldots$
p	T	T	F	T	T	F	T	T	T	T	$\ldots$
q	T	T	F	F	T	T	F	F	T	T	$\ldots$
$\Box\, p$											
$\Box\, q$											

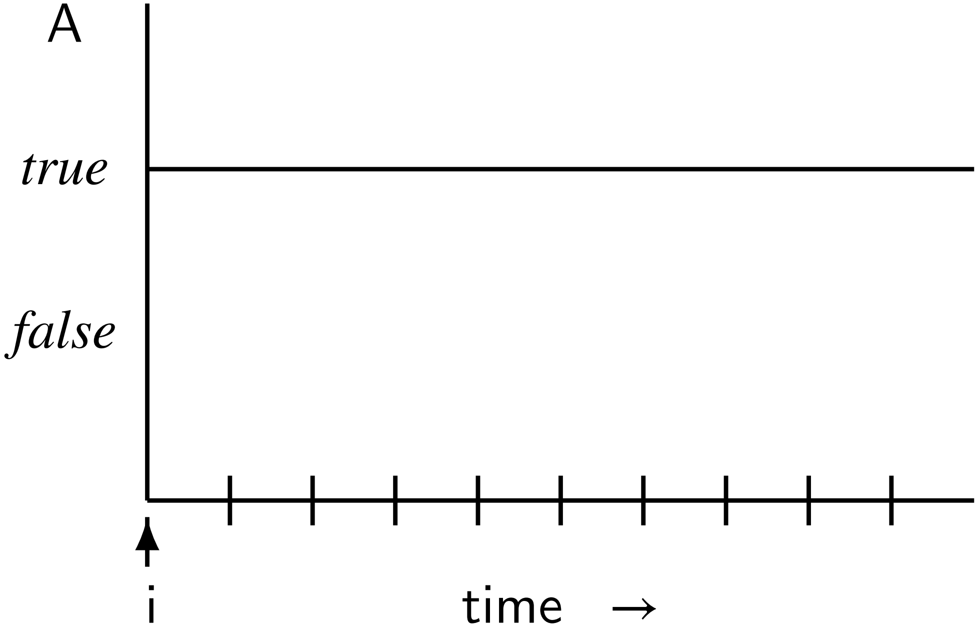
A Calculational Deductive System for Linear Temporal Logic

σ	s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	$\ldots$
p	T	T	F	T	T	F	T	T	T	T	$\ldots$
q	T	T	F	F	T	T	F	F	T	T	$\ldots$
$\Box p$	F	F	F	F	F	F	T	T	T	T	$\ldots$
$\Box q$											

A Calculational Deductive System for Linear Temporal Logic

σ	s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	$\ldots$
p	T	T	F	T	T	F	T	T	T	T	$\ldots$
q	T	T	F	F	T	T	F	F	T	T	$\ldots$
$\Box\, p$	F	F	F	F	F	F	T	T	T	T	$\ldots$
$\Box\, q$	F	F	F	F	F	F	F	F	F	F	$\ldots$

$\Box A$



$\Box A$ is a safety property.

Example: $\Box \neg(p4 \wedge q4)$

Algorithm 4.1: Third attempt

boolean wantp ← false, wantq ← false

p	q
loop forever	loop forever
p1: non-critical section	q1: non-critical section
p2: wantp ← true	q2: wantq ← true
p3: await wantq = false	q3: await wantp = false
p4: critical section	q4: critical section
p5: wantp ← false	q5: wantq ← false

To show starvation-free, must prove

$$\Box\,(p2 \Rightarrow \Diamond\, p4)$$

Algorithm 4.1: Third attempt

boolean wantp ← false, wantq ← false

p	**q**
loop forever	loop forever
p1: non-critical section	q1: non-critical section
p2: wantp ← true	q2: wantq ← true
p3: await wantq = false	q3: await wantp = false
p4: critical section	q4: critical section
p5: wantp ← false	q5: wantq ← false

True and False are constants

σ	s_0	s_1	s_2	s_3	s_4	$\ldots$
$true$	T	T	T	T	T	$\ldots$
$false$	F	F	F	F	F	$\ldots$

True and False are constants

σ	s_0	s_1	s_2	s_3	s_4	$\ldots$
$true$	T	T	T	T	T	$\ldots$
$false$	F	F	F	F	F	$\ldots$

The case analysis metatheorem is NOT valid in linear temporal logic!

A Calculational Deductive System for Linear Temporal Logic

Next $\bigcirc$

(1) **Axiom, Self-dual:** $\bigcirc \neg p \equiv \neg \bigcirc p$

(2) **Axiom, Distributivity of $\bigcirc$ over $\Rightarrow$:** $\bigcirc (p \Rightarrow q) \equiv \bigcirc p \Rightarrow \bigcirc q$

(3) **Linearity:** $\bigcirc p \equiv \neg \bigcirc \neg p$

A Calculational Deductive System for Linear Temporal Logic

(4) **Distributivity of $\circ$ over $\vee$:** $\circ(p \vee q) \equiv \circ p \vee \circ q$

Proof:

$$\circ(p \vee q)$$

A Calculational Deductive System for Linear Temporal Logic

(4) **Distributivity of $\bigcirc$ over $\vee$:** $\bigcirc(p \vee q) \equiv \bigcirc p \vee \bigcirc q$

Proof:

$$
\begin{array}{ll}
 & \bigcirc(p \vee q) \\
= & \langle \text{(3.59) Implication } p \Rightarrow q \equiv \neg p \vee q \rangle
\end{array}
$$

A Calculational Deductive System for Linear Temporal Logic

(4) **Distributivity of $\bigcirc$ over $\vee$:** $\bigcirc(p \vee q) \equiv \bigcirc p \vee \bigcirc q$

Proof:

$$\begin{array}{ll} & \bigcirc(p \vee q) \\ = & \langle (3.59)\ \text{Implication}\ p \Rightarrow q \ \equiv\ \neg p \vee q \rangle \\ & \bigcirc(\neg p \Rightarrow q) \end{array}$$

A Calculational Deductive System for Linear Temporal Logic

(4) **Distributivity of $\bigcirc$ over $\vee$:** $\bigcirc(p \vee q) \equiv \bigcirc p \vee \bigcirc q$

Proof:

$$
\begin{array}{ll}
 & \bigcirc(p \vee q) \\
= & \langle \text{(3.59) Implication } p \Rightarrow q \;\equiv\; \neg p \vee q \rangle \\
 & \bigcirc(\neg p \Rightarrow q) \\
= & \langle \text{(2) Distributivity of } \bigcirc \text{ over } \Rightarrow \rangle
\end{array}
$$

A Calculational Deductive System for Linear Temporal Logic

(4) **Distributivity of $\bigcirc$ over $\vee$:** $\bigcirc(p \vee q) \equiv \bigcirc p \vee \bigcirc q$

Proof:

$$
\begin{array}{ll}
 & \bigcirc(p \vee q) \\
= & \langle (3.59)\ \text{Implication } p \Rightarrow q \ \equiv\ \neg p \vee q \rangle \\
 & \bigcirc(\neg p \Rightarrow q) \\
= & \langle (2)\ \text{Distributivity of } \bigcirc \text{ over } \Rightarrow \rangle \\
 & \bigcirc \neg p \Rightarrow \bigcirc q
\end{array}
$$

A Calculational Deductive System for Linear Temporal Logic

(4) **Distributivity of $\circ$ over $\vee$:** $\circ(p \vee q) \equiv \circ p \vee \circ q$

Proof:

$$\begin{array}{ll}
 & \circ(p \vee q) \\
= & \langle (3.59) \text{ Implication } p \Rightarrow q \ \equiv\ \neg p \vee q \rangle \\
 & \circ(\neg p \Rightarrow q) \\
= & \langle (2) \text{ Distributivity of } \circ \text{ over } \Rightarrow \rangle \\
 & \circ \neg p \Rightarrow \circ q \\
= & \langle (3.59) \text{ Implication } p \Rightarrow q \ \equiv\ \neg p \vee q \text{ with } p, q := \circ \neg p, \circ q \rangle
\end{array}$$

A Calculational Deductive System for Linear Temporal Logic

(4) **Distributivity of $\bigcirc$ over $\vee$:** $\bigcirc(p \vee q) \equiv \bigcirc p \vee \bigcirc q$

Proof:

$$
\begin{array}{ll}
 & \bigcirc(p \vee q) \\
= & \langle \text{(3.59) Implication } p \Rightarrow q \ \equiv \ \neg p \vee q \rangle \\
 & \bigcirc(\neg p \Rightarrow q) \\
= & \langle \text{(2) Distributivity of } \bigcirc \text{ over } \Rightarrow \rangle \\
 & \bigcirc \neg p \Rightarrow \bigcirc q \\
= & \langle \text{(3.59) Implication } p \Rightarrow q \ \equiv \ \neg p \vee q \text{ with } p, q := \bigcirc \neg p, \bigcirc q \rangle \\
 & \neg \bigcirc \neg p \vee \bigcirc q
\end{array}
$$

A Calculational Deductive System for Linear Temporal Logic

(4) **Distributivity of $\bigcirc$ over $\vee$:** $\bigcirc(p \vee q) \equiv \bigcirc p \vee \bigcirc q$

Proof:

$$
\begin{array}{cl}
 & \bigcirc(p \vee q) \\
= & \langle \text{(3.59) Implication } p \Rightarrow q \;\equiv\; \neg p \vee q \rangle \\
 & \bigcirc(\neg p \Rightarrow q) \\
= & \langle \text{(2) Distributivity of } \bigcirc \text{ over } \Rightarrow \rangle \\
 & \bigcirc \neg p \Rightarrow \bigcirc q \\
= & \langle \text{(3.59) Implication } p \Rightarrow q \;\equiv\; \neg p \vee q \text{ with } p, q := \bigcirc \neg p, \bigcirc q \rangle \\
 & \neg \bigcirc \neg p \vee \bigcirc q \\
= & \langle \text{(3) Linearity} \rangle
\end{array}
$$

A Calculational Deductive System for Linear Temporal Logic

(4) **Distributivity of $\bigcirc$ over $\vee$:** $\bigcirc(p \vee q) \equiv \bigcirc p \vee \bigcirc q$

Proof:

$\bigcirc(p \vee q)$

$=$ ⟨(3.59) Implication $p \Rightarrow q \equiv \neg p \vee q$⟩

$\bigcirc(\neg p \Rightarrow q)$

$=$ ⟨(2) Distributivity of $\bigcirc$ over $\Rightarrow$⟩

$\bigcirc \neg p \Rightarrow \bigcirc q$

$=$ ⟨(3.59) Implication $p \Rightarrow q \equiv \neg p \vee q$ with $p, q := \bigcirc \neg p, \bigcirc q$⟩

$\neg \bigcirc \neg p \vee \bigcirc q$

$=$ ⟨(3) Linearity⟩

$\bigcirc p \vee \bigcirc q$ ▮

A Calculational Deductive System for Linear Temporal Logic

(5) **Distributivity of $\circ$ over $\wedge$:** $\circ(p \wedge q) \equiv \circ p \wedge \circ q$

Proof:

$$\circ(p \wedge q)$$

A Calculational Deductive System for Linear Temporal Logic

(5) **Distributivity of $\bigcirc$ over $\wedge$:** $\bigcirc(p \wedge q) \equiv \bigcirc p \wedge \bigcirc q$

Proof:

$$
\begin{array}{ll}
& \bigcirc(p \wedge q) \\
= & \langle \text{(3.12) Double negation, } \neg\neg p \equiv p \text{, twice} \rangle
\end{array}
$$

A Calculational Deductive System for Linear Temporal Logic

(5) **Distributivity of $\circ$ over $\wedge$:** $\circ(p \wedge q) \equiv \circ p \wedge \circ q$

Proof:

$$
\begin{array}{ll}
 & \circ(p \wedge q) \\
= & \langle \text{(3.12) Double negation, } \neg\neg p \equiv p \text{, twice} \rangle \\
 & \circ(\neg\neg p \wedge \neg\neg q)
\end{array}
$$

A Calculational Deductive System for Linear Temporal Logic

(5) **Distributivity of $\bigcirc$ over $\wedge$:** $\bigcirc(p \wedge q) \equiv \bigcirc p \wedge \bigcirc q$

Proof:

$$
\begin{array}{cl}
 & \bigcirc(p \wedge q) \\
= & \langle \text{(3.12) Double negation, } \neg\neg p \equiv p \text{, twice} \rangle \\
 & \bigcirc(\neg\neg p \wedge \neg\neg q) \\
= & \langle \text{(3.47b) De Morgan, } \neg(p \vee q) \equiv \neg p \wedge \neg q \rangle
\end{array}
$$

A Calculational Deductive System for Linear Temporal Logic

(5) **Distributivity of $\circ$ over $\wedge$:** $\circ(p \wedge q) \equiv \circ p \wedge \circ q$

Proof:

$$
\begin{array}{ll}
 & \circ(p \wedge q) \\
= & \langle \text{(3.12) Double negation, } \neg\neg p \equiv p\text{, twice} \rangle \\
 & \circ(\neg\neg p \wedge \neg\neg q) \\
= & \langle \text{(3.47b) De Morgan, } \neg(p \vee q) \equiv \neg p \wedge \neg q \rangle \\
 & \circ\neg(\neg p \vee \neg q)
\end{array}
$$

A Calculational Deductive System for Linear Temporal Logic

(5) **Distributivity of $\circ$ over $\wedge$:** $\circ\,(p \wedge q) \equiv \circ\, p \wedge \circ\, q$

Proof:

$$
\begin{array}{ll}
 & \circ\,(p \wedge q) \\
= & \langle \text{(3.12) Double negation, } \neg\neg p \equiv p \text{, twice} \rangle \\
 & \circ\,(\neg\neg p \wedge \neg\neg q) \\
= & \langle \text{(3.47b) De Morgan, } \neg(p \vee q) \equiv \neg p \wedge \neg q \rangle \\
 & \circ\,\neg(\neg p \vee \neg q) \\
= & \langle \text{(1) Self-dual with } p := (\neg p \vee \neg q) \rangle
\end{array}
$$

A Calculational Deductive System for Linear Temporal Logic

(5) **Distributivity of $\circ$ over $\wedge$:** $\circ(p \wedge q) \equiv \circ p \wedge \circ q$

Proof:

$$
\begin{array}{ll}
 & \circ(p \wedge q) \\
= & \langle \text{(3.12) Double negation, } \neg\neg p \equiv p \text{, twice} \rangle \\
 & \circ(\neg\neg p \wedge \neg\neg q) \\
= & \langle \text{(3.47b) De Morgan, } \neg(p \vee q) \equiv \neg p \wedge \neg q \rangle \\
 & \circ\neg(\neg p \vee \neg q) \\
= & \langle \text{(1) Self-dual with } p := (\neg p \vee \neg q) \rangle \\
 & \neg\circ(\neg p \vee \neg q)
\end{array}
$$

A Calculational Deductive System for Linear Temporal Logic

(5) **Distributivity of $\bigcirc$ over $\land$:** $\bigcirc(p \land q) \equiv \bigcirc p \land \bigcirc q$

Proof:

$$
\begin{array}{ll}
 & \bigcirc(p \land q) \\
= & \langle (3.12)\ \text{Double negation}, \neg\neg p \equiv p, \text{twice} \rangle \\
 & \bigcirc(\neg\neg p \land \neg\neg q) \\
= & \langle (3.47\text{b})\ \text{De Morgan}, \neg(p \lor q) \equiv \neg p \land \neg q \rangle \\
 & \bigcirc\neg(\neg p \lor \neg q) \\
= & \langle (1)\ \text{Self-dual with } p := (\neg p \lor \neg q) \rangle \\
 & \neg\bigcirc(\neg p \lor \neg q) \\
= & \langle (4)\ \text{Distributivity of } \bigcirc \text{ over } \lor \text{ with } p, q := \neg p, \neg q \rangle
\end{array}
$$

A Calculational Deductive System for Linear Temporal Logic

(5) **Distributivity of $\circ$ over $\wedge$:** $\circ (p \wedge q) \equiv \circ p \wedge \circ q$

Proof:

$$
\begin{array}{ll}
 & \circ (p \wedge q) \\
= & \langle \text{(3.12) Double negation, } \neg\neg p \equiv p, \text{ twice} \rangle \\
 & \circ (\neg\neg p \wedge \neg\neg q) \\
= & \langle \text{(3.47b) De Morgan, } \neg(p \vee q) \equiv \neg p \wedge \neg q \rangle \\
 & \circ \neg(\neg p \vee \neg q) \\
= & \langle \text{(1) Self-dual with } p := (\neg p \vee \neg q) \rangle \\
 & \neg\circ (\neg p \vee \neg q) \\
= & \langle \text{(4) Distributivity of } \circ \text{ over } \vee \text{ with } p, q := \neg p, \neg q \rangle \\
 & \neg(\circ \neg p \vee \circ \neg q)
\end{array}
$$

A Calculational Deductive System for Linear Temporal Logic

(5) **Distributivity of $\bigcirc$ over $\wedge$:** $\bigcirc(p \wedge q) \equiv \bigcirc p \wedge \bigcirc q$

Proof:

$$
\begin{array}{ll}
 & \bigcirc(p \wedge q) \\
= & \langle \text{(3.12) Double negation, } \neg\neg p \equiv p\text{, twice} \rangle \\
 & \bigcirc(\neg\neg p \wedge \neg\neg q) \\
= & \langle \text{(3.47b) De Morgan, } \neg(p \vee q) \equiv \neg p \wedge \neg q \rangle \\
 & \bigcirc\neg(\neg p \vee \neg q) \\
= & \langle \text{(1) Self-dual with } p := (\neg p \vee \neg q) \rangle \\
 & \neg\bigcirc(\neg p \vee \neg q) \\
= & \langle \text{(4) Distributivity of } \bigcirc \text{ over } \vee \text{ with } p, q := \neg p, \neg q \rangle \\
 & \neg(\bigcirc\neg p \vee \bigcirc\neg q) \\
= & \langle \text{(3.47b) De Morgan, } \neg(p \vee q) \equiv \neg p \wedge \neg q \rangle
\end{array}
$$

A Calculational Deductive System for Linear Temporal Logic

(5) **Distributivity of $\bigcirc$ over $\wedge$:** $\bigcirc(p \wedge q) \equiv \bigcirc p \wedge \bigcirc q$

Proof:

$$
\begin{array}{ll}
 & \bigcirc(p \wedge q) \\
= & \langle \text{(3.12) Double negation, } \neg\neg p \equiv p \text{, twice} \rangle \\
 & \bigcirc(\neg\neg p \wedge \neg\neg q) \\
= & \langle \text{(3.47b) De Morgan, } \neg(p \vee q) \equiv \neg p \wedge \neg q \rangle \\
 & \bigcirc \neg(\neg p \vee \neg q) \\
= & \langle \text{(1) Self-dual with } p := (\neg p \vee \neg q) \rangle \\
 & \neg\bigcirc(\neg p \vee \neg q) \\
= & \langle \text{(4) Distributivity of } \bigcirc \text{ over } \vee \text{ with } p, q := \neg p, \neg q \rangle \\
 & \neg(\bigcirc\neg p \vee \bigcirc\neg q) \\
= & \langle \text{(3.47b) De Morgan, } \neg(p \vee q) \equiv \neg p \wedge \neg q \rangle \\
 & \neg\bigcirc\neg p \wedge \neg\bigcirc\neg q
\end{array}
$$

A Calculational Deductive System for Linear Temporal Logic

(5) **Distributivity of $\circ$ over $\wedge$:** $\circ(p \wedge q) \equiv \circ p \wedge \circ q$

Proof:

$$
\begin{array}{ll}
 & \circ(p \wedge q) \\
= & \langle \text{(3.12) Double negation, } \neg\neg p \equiv p\text{, twice} \rangle \\
 & \circ(\neg\neg p \wedge \neg\neg q) \\
= & \langle \text{(3.47b) De Morgan, } \neg(p \vee q) \equiv \neg p \wedge \neg q \rangle \\
 & \circ\neg(\neg p \vee \neg q) \\
= & \langle \text{(1) Self-dual with } p := (\neg p \vee \neg q) \rangle \\
 & \neg\circ(\neg p \vee \neg q) \\
= & \langle \text{(4) Distributivity of } \circ \text{ over } \vee \text{ with } p, q := \neg p, \neg q \rangle \\
 & \neg(\circ\neg p \vee \circ\neg q) \\
= & \langle \text{(3.47b) De Morgan, } \neg(p \vee q) \equiv \neg p \wedge \neg q \rangle \\
 & \neg\circ\neg p \wedge \neg\circ\neg q \\
= & \langle \text{(3) Linearity, twice} \rangle
\end{array}
$$

(5) **Distributivity of $\bigcirc$ over $\wedge$:** $\bigcirc(p \wedge q) \equiv \bigcirc p \wedge \bigcirc q$

Proof:

$$
\begin{array}{cl}
 & \bigcirc(p \wedge q) \\
= & \langle \text{(3.12) Double negation, } \neg\neg p \equiv p \text{, twice} \rangle \\
 & \bigcirc(\neg\neg p \wedge \neg\neg q) \\
= & \langle \text{(3.47b) De Morgan, } \neg(p \vee q) \equiv \neg p \wedge \neg q \rangle \\
 & \bigcirc\neg(\neg p \vee \neg q) \\
= & \langle \text{(1) Self-dual with } p := (\neg p \vee \neg q) \rangle \\
 & \neg\bigcirc(\neg p \vee \neg q) \\
= & \langle \text{(4) Distributivity of } \bigcirc \text{ over } \vee \text{ with } p, q := \neg p, \neg q \rangle \\
 & \neg(\bigcirc\neg p \vee \bigcirc\neg q) \\
= & \langle \text{(3.47b) De Morgan, } \neg(p \vee q) \equiv \neg p \wedge \neg q \rangle \\
 & \neg\bigcirc\neg p \wedge \neg\bigcirc\neg q \\
= & \langle \text{(3) Linearity, twice} \rangle \\
 & \bigcirc p \wedge \bigcirc q \quad \blacksquare
\end{array}
$$

A Calculational Deductive System for Linear Temporal Logic

(6) **Distributivity of $\bigcirc$ over $\equiv$:** $\bigcirc\,(p \equiv q) \equiv \bigcirc\,p \equiv \bigcirc\,q$

Proof:
Exercise for the student. Hint: Start with mutual implication.

A Calculational Deductive System for Linear Temporal Logic

(7) **Truth of $\bigcirc$:** $\bigcirc\, true \equiv true$

Proof:

$$\bigcirc\, true$$

A Calculational Deductive System for Linear Temporal Logic

(7) **Truth of $\bigcirc$:** $\bigcirc\, true \equiv true$

Proof:

$$
\begin{array}{ll}
 & \bigcirc\, true \\
= & \langle (3.28)\ \text{Excluded middle}\ p \vee \neg p \rangle
\end{array}
$$

A Calculational Deductive System for Linear Temporal Logic

(7) **Truth of $\bigcirc$:** $\bigcirc\, true \equiv true$

Proof:

$$\begin{array}{ll} & \bigcirc\, true \\ = & \langle \text{(3.28) Excluded middle } p \vee \neg p \rangle \\ & \bigcirc\, (p \vee \neg p) \end{array}$$

A Calculational Deductive System for Linear Temporal Logic

(7) **Truth of $\bigcirc$:** $\bigcirc\, true \equiv true$

Proof:

$$
\begin{array}{ll}
 & \bigcirc\, true \\
= & \langle \text{(3.28) Excluded middle } p \vee \neg p \rangle \\
 & \bigcirc\, (p \vee \neg p) \\
= & \langle \text{(4) Distributivty of } \bigcirc \text{ over } \vee \rangle
\end{array}
$$

A Calculational Deductive System for Linear Temporal Logic

(7) **Truth of $\bigcirc$:** $\bigcirc\, true \equiv true$

Proof:

$$
\begin{array}{ll}
 & \bigcirc\, true \\
= & \langle \text{(3.28) Excluded middle } p \vee \neg p \rangle \\
 & \bigcirc\, (p \vee \neg p) \\
= & \langle \text{(4) Distributivty of } \bigcirc \text{ over } \vee \rangle \\
 & \bigcirc\, p \vee \bigcirc\, \neg p
\end{array}
$$

A Calculational Deductive System for Linear Temporal Logic

(7) **Truth of $\bigcirc$:** $\bigcirc\, true \equiv true$

Proof:

$$
\begin{array}{ll}
 & \bigcirc\, true \\
= & \langle \text{(3.28) Excluded middle } p \vee \neg p \rangle \\
 & \bigcirc\, (p \vee \neg p) \\
= & \langle \text{(4) Distributivty of } \bigcirc \text{ over } \vee \rangle \\
 & \bigcirc\, p \vee \bigcirc \neg p \\
= & \langle \text{(1) Self-dual} \rangle
\end{array}
$$

A Calculational Deductive System for Linear Temporal Logic

(7) **Truth of $\bigcirc$:** $\bigcirc\, true \equiv true$

Proof:

$$\begin{array}{ll}
 & \bigcirc\, true \\
= & \langle \text{(3.28) Excluded middle } p \vee \neg p \rangle \\
 & \bigcirc\, (p \vee \neg p) \\
= & \langle \text{(4) Distributivty of } \bigcirc \text{ over } \vee \rangle \\
 & \bigcirc\, p \vee \bigcirc\, \neg p \\
= & \langle \text{(1) Self-dual} \rangle \\
 & \bigcirc\, p \vee \neg \bigcirc\, p
\end{array}$$

A Calculational Deductive System for Linear Temporal Logic

(7) **Truth of $\bigcirc$:** $\bigcirc\, true \equiv true$

Proof:

$\bigcirc\, true$

$=$ $\langle$(3.28) Excluded middle $p \vee \neg p\rangle$

$\bigcirc\, (p \vee \neg p)$

$=$ $\langle$(4) Distributivty of $\bigcirc$ over $\vee\rangle$

$\bigcirc\, p \vee \bigcirc\, \neg p$

$=$ $\langle$(1) Self-dual$\rangle$

$\bigcirc\, p \vee \neg \bigcirc\, p$

$=$ $\langle$(3.28) Excluded middle $p \vee \neg p$ with $p := \bigcirc\, p\rangle$

A Calculational Deductive System for Linear Temporal Logic

(7) **Truth of $\bigcirc$:** $\bigcirc\, true \equiv true$

Proof:

$$
\begin{array}{ll}
 & \bigcirc\, true \\
= & \langle (3.28)\text{ Excluded middle } p \vee \neg p \rangle \\
 & \bigcirc\, (p \vee \neg p) \\
= & \langle (4)\text{ Distributivty of } \bigcirc \text{ over } \vee \rangle \\
 & \bigcirc\, p \vee \bigcirc\, \neg p \\
= & \langle (1)\text{ Self-dual} \rangle \\
 & \bigcirc\, p \vee \neg \bigcirc\, p \\
= & \langle (3.28)\text{ Excluded middle } p \vee \neg p \text{ with } p := \bigcirc\, p \rangle \\
 & true
\end{array}
$$

A Calculational Deductive System for Linear Temporal Logic

(8) **Falsehood of $\bigcirc$:** $\bigcirc\, false \equiv false$

Proof:
Exercise for the student.

A Calculational Deductive System for Linear Temporal Logic

Until $\mathcal{U}$

(9) **Axiom, Distributivity of $\circ$ over $\mathcal{U}$:** $\circ (p \, \mathcal{U} \, q) \equiv \circ p \, \mathcal{U} \circ q$

(10) **Axiom, Expansion of $\mathcal{U}$:** $p \, \mathcal{U} \, q \equiv q \vee (p \wedge \circ (p \, \mathcal{U} \, q))$

(11) **Axiom, Right zero of $\mathcal{U}$:** $p \, \mathcal{U} \, false \equiv false$

(12) **Axiom, Left distributivity of $\mathcal{U}$ over $\vee$:** $p \, \mathcal{U} \, (q \vee r) \equiv p \, \mathcal{U} \, q \vee p \, \mathcal{U} \, r$

(13) **Axiom, Right distributivity of $\mathcal{U}$ over $\vee$:** $p \, \mathcal{U} \, r \vee q \, \mathcal{U} \, r \Rightarrow (p \vee q) \, \mathcal{U} \, r$

(14) **Axiom, Left distributivity of $\mathcal{U}$ over $\wedge$:** $p \, \mathcal{U} \, (q \wedge r) \Rightarrow p \, \mathcal{U} \, q \wedge p \, \mathcal{U} \, r$

(15) **Axiom, Right distributivity of $\mathcal{U}$ over $\wedge$:** $(p \wedge q) \, \mathcal{U} \, r \equiv p \, \mathcal{U} \, r \wedge q \, \mathcal{U} \, r$

(16) **Axiom, $\mathcal{U}$ implication ordering:** $p \, \mathcal{U} \, q \wedge \neg q \, \mathcal{U} \, r \Rightarrow p \, \mathcal{U} \, r$

(17) **Axiom, Right $\mathcal{U} \vee$ ordering:** $p \, \mathcal{U} \, (q \, \mathcal{U} \, r) \Rightarrow (p \vee q) \, \mathcal{U} \, r$

(18) **Axiom, Right $\wedge \mathcal{U}$ ordering:** $p \, \mathcal{U} \, (q \wedge r) \Rightarrow (p \, \mathcal{U} \, q) \, \mathcal{U} \, r$

A Calculational Deductive System for Linear Temporal Logic

(19) **Right distributivity of $\mathcal{U}$ over $\Rightarrow$:** $(p \Rightarrow q)\ \mathcal{U}\ r \Rightarrow (p\ \mathcal{U}\ r \Rightarrow q\ \mathcal{U}\ r)$

(20) **Right zero of $\mathcal{U}$:** $p\ \mathcal{U}\ true \equiv true$

(21) **Left identity of $\mathcal{U}$:** $false\ \mathcal{U}\ q \equiv q$

(22) **Idempotency of $\mathcal{U}$:** $p\ \mathcal{U}\ p \equiv p$

(23) **$\mathcal{U}$ excluded middle:** $p\ \mathcal{U}\ q \vee p\ \mathcal{U}\ \neg q$

(24) $\neg p\ \mathcal{U}\ (q\ \mathcal{U}\ r) \wedge p\ \mathcal{U}\ r \Rightarrow q\ \mathcal{U}\ r$

A Calculational Deductive System for Linear Temporal Logic

(22) **Idempotency of $\mathcal{U}$:** $p \,\mathcal{U}\, p \equiv p$

Proof :

A Calculational Deductive System for Linear Temporal Logic

(22) **Idempotency of $\mathcal{U}$:** $p \,\mathcal{U}\, p \equiv p$

Proof:

$$p \,\mathcal{U}\, p$$

A Calculational Deductive System for Linear Temporal Logic

(22) **Idempotency of $\mathcal{U}$:** $p \,\mathcal{U}\, p \equiv p$

Proof:

$$
\begin{array}{ll}
 & p \,\mathcal{U}\, p \\
= & \langle \text{(10) Expansion of } \mathcal{U} \rangle
\end{array}
$$

A Calculational Deductive System for Linear Temporal Logic

(22) **Idempotency of $\mathcal{U}$:** $p \, \mathcal{U} \, p \equiv p$

Proof:

$$
\begin{aligned}
& p \, \mathcal{U} \, p \\
= & \quad \langle \text{(10) Expansion of } \mathcal{U} \rangle \\
& p \vee (p \wedge \circ \, (p \, \mathcal{U} \, p))
\end{aligned}
$$

A Calculational Deductive System for Linear Temporal Logic

(22) **Idempotency of $\mathcal{U}$:** $p \,\mathcal{U}\, p \equiv p$

Proof :

$$\begin{array}{ll}
& p \,\mathcal{U}\, p \\
= & \langle \text{(10) Expansion of } \mathcal{U} \rangle \\
& p \vee (p \wedge \circ\, (p \,\mathcal{U}\, p)) \\
= & \langle \text{(3.43b) Absorption}, \; p \vee (p \wedge q) \equiv p \text{ with } q := \circ\, (p \,\mathcal{U}\, p) \rangle
\end{array}$$

A Calculational Deductive System for Linear Temporal Logic

(22) **Idempotency of $\mathcal{U}$:** $p \, \mathcal{U} \, p \equiv p$

Proof:

$$
\begin{array}{ll}
 & p \, \mathcal{U} \, p \\
= & \langle \text{(10) Expansion of } \mathcal{U} \rangle \\
 & p \vee (p \wedge \circ \, (p \, \mathcal{U} \, p)) \\
= & \langle \text{(3.43b) Absorption}, \; p \vee (p \wedge q) \equiv p \text{ with } q := \circ \, (p \, \mathcal{U} \, p) \rangle \\
 & p \quad \blacksquare
\end{array}
$$

A Calculational Deductive System for Linear Temporal Logic

(25) $p \,\mathcal{U}\, (\neg q \,\mathcal{U}\, r) \wedge q \,\mathcal{U}\, r \Rightarrow p \,\mathcal{U}\, r$

(26) $p \,\mathcal{U}\, q \wedge \neg q \,\mathcal{U}\, p \Rightarrow p$

(27) $p \wedge \neg p \,\mathcal{U}\, q \Rightarrow q$

(28) $p \,\mathcal{U}\, q \Rightarrow p \vee q$

(29) $\mathcal{U}$ **insertion:** $q \Rightarrow p \,\mathcal{U}\, q$

(30) $p \wedge q \Rightarrow p \,\mathcal{U}\, q$

A Calculational Deductive System for Linear Temporal Logic

(29) $\mathcal{U}$ **Insertion:** $q \Rightarrow p \, \mathcal{U} \, q$

Proof:

A Calculational Deductive System for Linear Temporal Logic

(29) $\mathcal{U}$ **Insertion:** $q \Rightarrow p \, \mathcal{U} \, q$

Proof:

$$p \, \mathcal{U} \, q$$

A Calculational Deductive System for Linear Temporal Logic

(29) $\mathcal{U}$ **Insertion:** $q \Rightarrow p \,\mathcal{U}\, q$

Proof:

$$
\begin{array}{ll}
 & p \,\mathcal{U}\, q \\
= & \langle \text{(10) Expansion of } \mathcal{U} \rangle
\end{array}
$$

A Calculational Deductive System for Linear Temporal Logic

(29) $\mathcal{U}$ **Insertion:** $q \Rightarrow p \,\mathcal{U}\, q$

Proof:

$$
\begin{array}{ll}
& p \,\mathcal{U}\, q \\
= & \langle \text{(10) Expansion of } \mathcal{U} \,\rangle \\
& q \vee (p \wedge \circ\, (p \,\mathcal{U}\, q))
\end{array}
$$

(29) $\mathcal{U}$ **Insertion:** $q \Rightarrow p \,\mathcal{U}\, q$

Proof:

$$\begin{array}{ll} & p \,\mathcal{U}\, q \\ = & \langle \text{(10) Expansion of } \mathcal{U} \rangle \\ & q \vee (p \wedge \circ\, (p \,\mathcal{U}\, q)) \\ \Leftarrow & \langle \text{(3.76a) Weakening}, \, p \Rightarrow p \vee q \rangle \end{array}$$

(29) $\mathcal{U}$ **Insertion:** $q \Rightarrow p \, \mathcal{U} \, q$

Proof:

$$
\begin{array}{ll}
 & p \, \mathcal{U} \, q \\
= & \quad \langle \text{(10) Expansion of } \mathcal{U} \rangle \\
 & q \vee (p \wedge \circ \, (p \, \mathcal{U} \, q)) \\
\Leftarrow & \quad \langle \text{(3.76a) Weakening}, p \Rightarrow p \vee q \rangle \\
 & q \quad \blacksquare
\end{array}
$$

A Calculational Deductive System for Linear Temporal Logic

(30) $p \wedge q \Rightarrow p \,\mathcal{U}\, q$

(31) **Absorption:** $p \vee p \,\mathcal{U}\, q \equiv p \vee q$

(32) **Absorption:** $p \,\mathcal{U}\, q \vee q \equiv p \,\mathcal{U}\, q$

(33) **Absorption:** $p \,\mathcal{U}\, q \wedge q \equiv q$

(34) **Absorption:** $p \,\mathcal{U}\, q \vee (p \wedge q) \equiv p \,\mathcal{U}\, q$

(35) **Absorption:** $p \,\mathcal{U}\, q \wedge (p \vee q) \equiv p \,\mathcal{U}\, q$

(36) **Left absorption of $\mathcal{U}$:** $p \,\mathcal{U}\, (p \,\mathcal{U}\, q) \equiv p \,\mathcal{U}\, q$

(37) **Right absorption of $\mathcal{U}$:** $(p \,\mathcal{U}\, q) \,\mathcal{U}\, q \equiv p \,\mathcal{U}\, q$

Eventually $\Diamond$

(38) **Definition of $\Diamond$:** $\Diamond q \equiv \mathit{true}\ \mathcal{U}\ q$

A Calculational Deductive System for Linear Temporal Logic

(39) **Absorption of $\Diamond$ into $\mathcal{U}$:** $p \,\mathcal{U}\, q \wedge \Diamond q \equiv p \,\mathcal{U}\, q$

(40) **Absorption of $\mathcal{U}$ into $\Diamond$:** $p \,\mathcal{U}\, q \vee \Diamond q \equiv \Diamond q$

(41) **Absorption of $\mathcal{U}$ into $\Diamond$:** $p \,\mathcal{U}\, \Diamond q \equiv \Diamond q$

(42) **Eventuality:** $p \,\mathcal{U}\, q \Rightarrow \Diamond q$

(43) **Truth of $\Diamond$:** $\Diamond\, true \equiv true$

(44) **Falsehood of $\Diamond$:** $\Diamond\, false \equiv false$

(45) **Expansion of $\Diamond$:** $\Diamond p \equiv p \vee \bigcirc \Diamond p$

(46) **Weakening of $\Diamond$:** $p \Rightarrow \Diamond p$

(47) **Weakening of $\Diamond$:** $\bigcirc p \Rightarrow \Diamond p$

A Calculational Deductive System for Linear Temporal Logic

(46) **Weakening of $\Diamond$:** $p \Rightarrow \Diamond p$

Proof:

$$\Diamond p$$

A Calculational Deductive System for Linear Temporal Logic

(46) **Weakening of $\Diamond$:** $p \Rightarrow \Diamond p$

Proof:

$$\begin{array}{ll} & \Diamond p \\ = & \langle \text{(45) Expansion of } \Diamond \rangle \end{array}$$

A Calculational Deductive System for Linear Temporal Logic

(46) **Weakening of $\Diamond$:** $p \Rightarrow \Diamond p$

Proof:

$$\begin{array}{ll} & \Diamond p \\ = & \langle (45)\ \text{Expansion of } \Diamond \rangle \\ & p \vee \bigcirc \Diamond p \end{array}$$

(46) **Weakening of $\Diamond$:** $p \Rightarrow \Diamond p$

Proof:

$$
\begin{array}{cl}
 & \Diamond p \\
= & \langle \text{(45) Expansion of } \Diamond \rangle \\
 & p \vee \bigcirc \Diamond p \\
\Leftarrow & \langle \text{(3.76a) Weakening the consequent, } p \Rightarrow p \vee q \rangle
\end{array}
$$

A Calculational Deductive System for Linear Temporal Logic

(46) **Weakening of $\Diamond$:** $p \Rightarrow \Diamond p$

Proof:

$$
\begin{array}{ll}
 & \Diamond p \\
= & \langle \text{(45) Expansion of } \Diamond \rangle \\
 & p \vee \bigcirc \Diamond p \\
\Leftarrow & \langle \text{(3.76a) Weakening the consequent, } p \Rightarrow p \vee q \rangle \\
 & p \quad \blacksquare
\end{array}
$$

A Calculational Deductive System for Linear Temporal Logic

(48) **Absorption of $\vee$ into $\Diamond$:** $p \vee \Diamond p \equiv \Diamond p$

(49) **Absorption of $\Diamond$ into $\wedge$:** $\Diamond p \wedge p \equiv p$

(50) **Absorption of $\Diamond$:** $\Diamond \Diamond p \equiv \Diamond p$

(51) **Exchange of $\bigcirc$ and $\Diamond$:** $\bigcirc \Diamond p \equiv \Diamond \bigcirc p$

(52) **Distributivity of $\Diamond$ over $\vee$:** $\Diamond (p \vee q) \equiv \Diamond p \vee \Diamond q$

(53) **Distributivity of $\Diamond$ over $\wedge$:** $\Diamond (p \wedge q) \Rightarrow \Diamond p \wedge \Diamond q$

A Calculational Deductive System for Linear Temporal Logic

Always $\Box$

(54) **Definition of $\Box$:** $\Box\, p \equiv \neg \Diamond \neg p$

(55) **Axiom, $\mathcal{U}$ Induction:** $\Box\,(p \Rightarrow (\bigcirc\, p \wedge q) \vee r) \Rightarrow (p \Rightarrow \Box\, q \vee q \,\mathcal{U}\, r)$

(56) **Axiom, $\mathcal{U}$ Induction:** $\Box\,(p \Rightarrow \bigcirc\,(p \vee q)) \Rightarrow (p \Rightarrow \Box\, p \vee p \,\mathcal{U}\, q)$

A Calculational Deductive System for Linear Temporal Logic

(57) **$\Box$ Induction:** $\Box(p \Rightarrow \bigcirc p) \Rightarrow (p \Rightarrow \Box p)$

(58) **$\Diamond$ Induction:** $\Box(\bigcirc p \Rightarrow p) \Rightarrow (\Diamond p \Rightarrow p)$

(59) $\Diamond p \equiv \neg \Box \neg p$

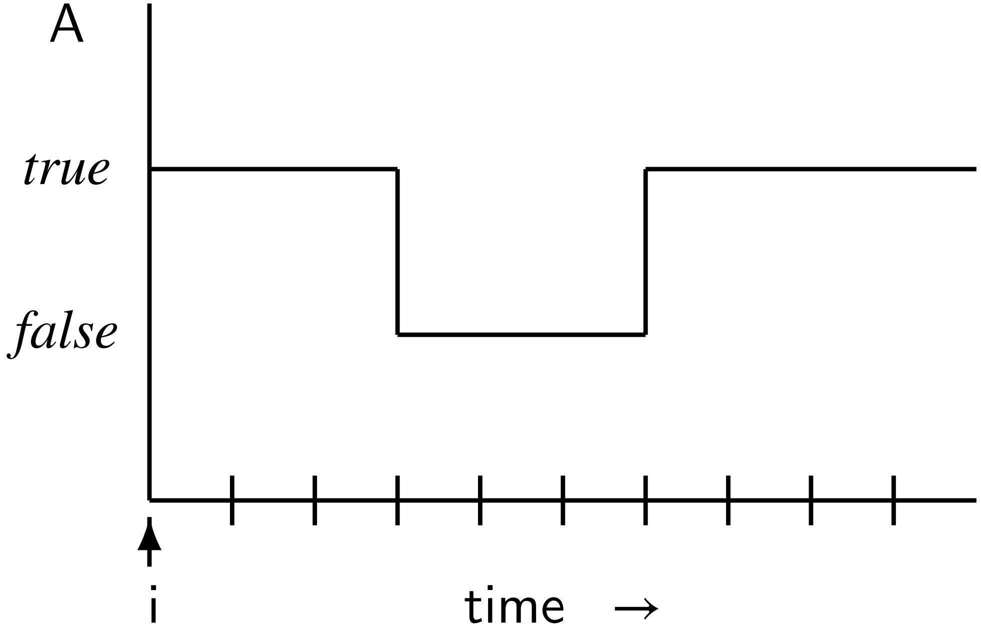
(60) **Dual of $\Box$:** $\neg \Box p \equiv \Diamond \neg p$

(61) **Dual of $\Diamond$:** $\neg \Diamond p \equiv \Box \neg p$

(62) **Dual of $\Diamond \Box$:** $\neg \Diamond \Box p \equiv \Box \Diamond \neg p$

(63) **Dual of $\Box \Diamond$:** $\neg \Box \Diamond p \equiv \Diamond \Box \neg p$

Duality: $\neg\Box A$



(60) **Dual of $\Box$:** $\neg\Box p \equiv \Diamond\neg p$

A Calculational Deductive System for Linear Temporal Logic

(61) **Dual of $\Diamond$:** $\neg\Diamond p \equiv \Box\neg p$

Proof:

$$\Box\neg p$$

A Calculational Deductive System for Linear Temporal Logic

(61) **Dual of $\Diamond$:** $\neg\Diamond p \equiv \Box\neg p$

Proof:

$$\begin{array}{ll} & \Box\neg p \\ = & \langle (54)\ \text{Definition of } \Box \rangle \end{array}$$

A Calculational Deductive System for Linear Temporal Logic

(61) **Dual of $\Diamond$:** $\neg\Diamond p \equiv \Box\neg p$

Proof:

$$\begin{array}{ll} & \Box\neg p \\ = & \langle \text{(54) Definition of } \Box \rangle \\ & \neg\Diamond\neg\neg p \end{array}$$

A Calculational Deductive System for Linear Temporal Logic

(61) **Dual of $\Diamond$:** $\neg\Diamond p \equiv \Box\neg p$

Proof:

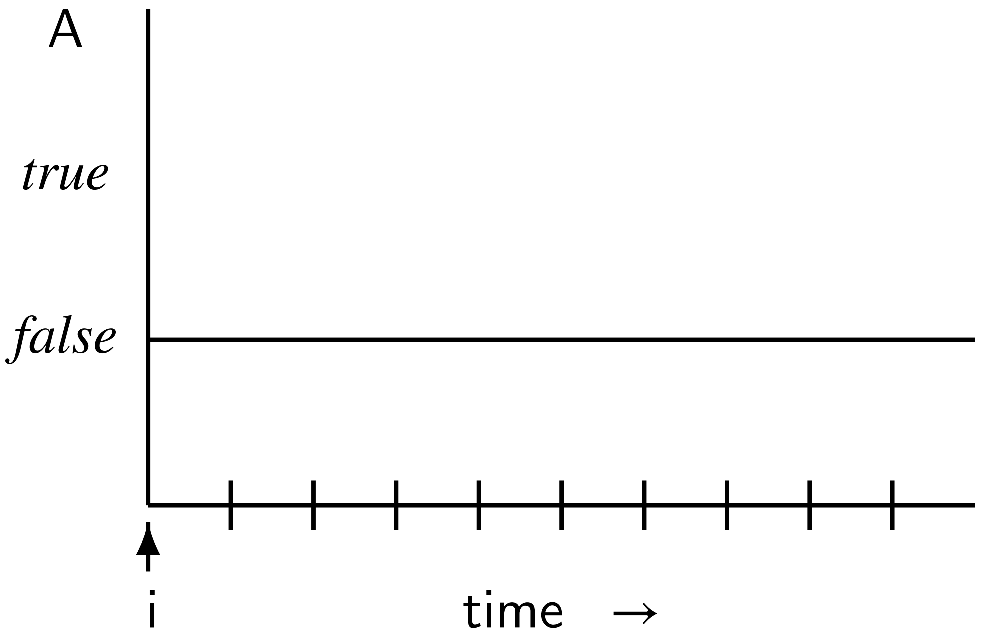
$$\begin{array}{ll}
 & \Box\neg p \\
= & \langle (54)\ \text{Definition of } \Box \rangle \\
 & \neg\Diamond\neg\neg p \\
= & \langle (3.12)\ \text{Double negation}, \neg\neg p \equiv p \rangle
\end{array}$$

(61) **Dual of $\Diamond$:** $\neg\Diamond p \equiv \Box\neg p$

Proof:

$$\begin{array}{ll} & \Box\neg p \\ = & \langle (54)\ \text{Definition of}\ \Box \rangle \\ & \neg\Diamond\neg\neg p \\ = & \langle (3.12)\ \text{Double negation},\ \neg\neg p \equiv p \rangle \\ & \neg\Diamond p \qquad \blacksquare \end{array}$$

Duality: $\neg \Diamond A$



$$\text{(61)} \quad \textbf{Dual of } \Diamond\textbf{:} \quad \neg\Diamond p \equiv \Box\neg p$$

A Calculational Deductive System for Linear Temporal Logic

(64) **Truth of $\Box$:** $\Box\, true \equiv true$

(65) **Falsehood of $\Box$:** $\Box\, false \equiv false$

(66) **Expansion of $\Box$:** $\Box\, p \equiv p \wedge \bigcirc \Box\, p$

(67) **Expansion of $\Box$:** $\Box\, p \equiv p \wedge \bigcirc p \wedge \bigcirc \Box\, p$

A Calculational Deductive System for Linear Temporal Logic

(68) **Absorption of $\wedge$ into $\Box$:** $p \wedge \Box p \equiv \Box p$

(69) **Absorption of $\Box$ into $\vee$:** $\Box p \vee p \equiv p$

(70) **Absorption of $\Diamond$ into $\Box$:** $\Diamond p \wedge \Box p \equiv \Box p$

(71) **Absorption of $\Box$ into $\Diamond$:** $\Box p \vee \Diamond p \equiv \Diamond p$

(72) **Absorption of $\Box$:** $\Box \Box p \equiv \Box p$

(73) **Exchange of $\bigcirc$ and $\Box$:** $\bigcirc \Box p \equiv \Box \bigcirc p$

(74) $p \Rightarrow \Box p \equiv p \Rightarrow \bigcirc \Box p$

A Calculational Deductive System for Linear Temporal Logic

(75) $p \wedge \Diamond \neg p \Rightarrow \Diamond (p \wedge \bigcirc \neg p)$

(76) **Strengthening of $\Box$:** $\Box\, p \Rightarrow p$

(77) **Strengthening of $\Box$:** $\Box\, p \Rightarrow \Diamond\, p$

(78) **Strengthening of $\Box$:** $\Box\, p \Rightarrow \bigcirc\, p$

(79) **Strengthening of $\Box$:** $\Box\, p \Rightarrow \bigcirc \Box\, p$

(80) $\bigcirc$ **generalization:** $\Box\, p \Rightarrow \Box \bigcirc p$

(81) $\Box\, p \Rightarrow \neg(q\ \mathcal{U}\ \neg p)$

Temporal deduction

(82) **Temporal deduction:**

To prove $\square P_1 \wedge \square P_2 \Rightarrow Q$, assume P_1 and P_2, and prove Q. You cannot use textual substitution in P_1 or P_2.

Always, continued

(83) **Distributivity of $\wedge$ over $\mathcal{U}$:** $\Box\, p \wedge q \;\mathcal{U}\; r \Rightarrow (p \wedge q) \;\mathcal{U}\; (p \wedge r)$

(84) **$\mathcal{U}$ implication:** $\Box\, p \wedge \Diamond\, q \Rightarrow p \;\mathcal{U}\; q$

(85) **Right monotonicity of $\mathcal{U}$:** $\Box\,(p \Rightarrow q) \Rightarrow (r \;\mathcal{U}\; p \Rightarrow r \;\mathcal{U}\; q)$

(86) **Left monotonicity of $\mathcal{U}$:** $\Box\,(p \Rightarrow q) \Rightarrow (p \;\mathcal{U}\; r \Rightarrow q \;\mathcal{U}\; r)$

(87) **Distributivity of $\neg$ over $\Box$:** $\Box\,\neg p \Rightarrow \neg\Box\, p$

(88) **Distributivity of $\Diamond$ over $\wedge$:** $\Box\, p \wedge \Diamond\, q \Rightarrow \Diamond\,(p \wedge q)$

A Calculational Deductive System for Linear Temporal Logic

(89) $\Diamond$ **excluded middle:** $\Diamond p \vee \Box \neg p$

(90) $\Box$ **excluded middle:** $\Box p \vee \Diamond \neg p$

(91) **Temporal excluded middle:** $\Diamond p \vee \Diamond \neg p$

(92) $\Diamond$ **contradiction:** $\Diamond p \wedge \Box \neg p \equiv false$

(93) $\Box$ **contradiction:** $\Box p \wedge \Diamond \neg p \equiv false$

(94) **Temporal contradiction:** $\Box p \wedge \Box \neg p \equiv false$

(95) $\Box \Diamond$ **excluded middle:** $\Box \Diamond p \vee \Diamond \Box \neg p$

(96) $\Diamond \Box$ **excluded middle:** $\Diamond \Box p \vee \Box \Diamond \neg p$

(97) $\Box \Diamond$ **contradiction:** $\Box \Diamond p \wedge \Diamond \Box \neg p \equiv false$

(98) $\Diamond \Box$ **contradiction:** $\Diamond \Box p \wedge \Box \Diamond \neg p \equiv false$

A Calculational Deductive System for Linear Temporal Logic

(99) **Distributivity of $\Box$ over $\wedge$:** $\Box(p \wedge q) \equiv \Box p \wedge \Box q$

(100) **Distributivity of $\Box$ over $\vee$:** $\Box p \vee \Box q \Rightarrow \Box(p \vee q)$

(101) **Logical equivalence law of $\bigcirc$:** $\Box(p \equiv q) \Rightarrow (\bigcirc p \equiv \bigcirc q)$

(102) **Logical equivalence law of $\Diamond$:** $\Box(p \equiv q) \Rightarrow (\Diamond p \equiv \Diamond q)$

(103) **Logical equivalence law of $\Box$:** $\Box(p \equiv q) \Rightarrow (\Box p \equiv \Box q)$

(104) **Distributivity of $\Diamond$ over $\Rightarrow$:** $\Diamond(p \Rightarrow q) \equiv (\Box p \Rightarrow \Diamond q)$

(105) **Distributivity of $\Diamond$ over $\Rightarrow$:** $(\Diamond p \Rightarrow \Diamond q) \Rightarrow \Diamond(p \Rightarrow q)$

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Proof metatheorems

(136) **Metatheorem:** P is a theorem iff $\Box P$ is a theorem.

(137) **Metatheorem $\circ$:** If $P \Rightarrow Q$ is a theorem then $\circ P \Rightarrow \circ Q$ is a theorem.

(138) **Metatheorem $\Diamond$:** If $P \Rightarrow Q$ is a theorem then $\Diamond P \Rightarrow \Diamond Q$ is a theorem.

(139) **Metatheorem $\Box$:** If $P \Rightarrow Q$ is a theorem then $\Box P \Rightarrow \Box Q$ is a theorem.

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(140) $\mathcal{U}\,\Box$ **implication:** $p\,\mathcal{U}\,\Box\,q \Rightarrow \Box\,(p\,\mathcal{U}\,q)$

(141) **Absorption of** $\mathcal{U}$ **into** $\Box$**:** $p\,\mathcal{U}\,\Box\,p \equiv \Box\,p$

(142) **Right** $\wedge\,\mathcal{U}$ **strengthening:** $p\,\mathcal{U}\,(q \wedge r) \Rightarrow p\,\mathcal{U}\,(q\,\mathcal{U}\,r)$

(143) **Left** $\wedge\,\mathcal{U}$ **strengthening:** $(p \wedge q)\,\mathcal{U}\,r \Rightarrow (p\,\mathcal{U}\,q)\,\mathcal{U}\,r$

(144) **Left** $\wedge\,\mathcal{U}$ **ordering:** $(p \wedge q)\,\mathcal{U}\,r \Rightarrow p\,\mathcal{U}\,(q\,\mathcal{U}\,r)$

(145) $\Diamond\,\Box$ **implication:** $\Diamond\,\Box\,p \Rightarrow \Box\,\Diamond\,p$

(146) $\Box\,\Diamond$ **excluded middle:** $\Box\,\Diamond\,p \vee \Box\,\Diamond\,\neg p$

(147) $\Diamond\,\Box$ **contradiction:** $\Diamond\,\Box\,p \wedge \Diamond\,\Box\,\neg p \;\equiv\; false$

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(151) **Absorption of $\Diamond$ into $\Box\Diamond$:** $\Diamond\Box\Diamond p \equiv \Box\Diamond p$

(152) **Absorption of $\Box$ into $\Diamond\Box$:** $\Box\Diamond\Box p \equiv \Diamond\Box p$

(153) **Absorption of $\Box\Diamond$:** $\Box\Diamond\Box\Diamond p \equiv \Box\Diamond p$

(154) **Absorption of $\Diamond\Box$:** $\Diamond\Box\Diamond\Box p \equiv \Diamond\Box p$

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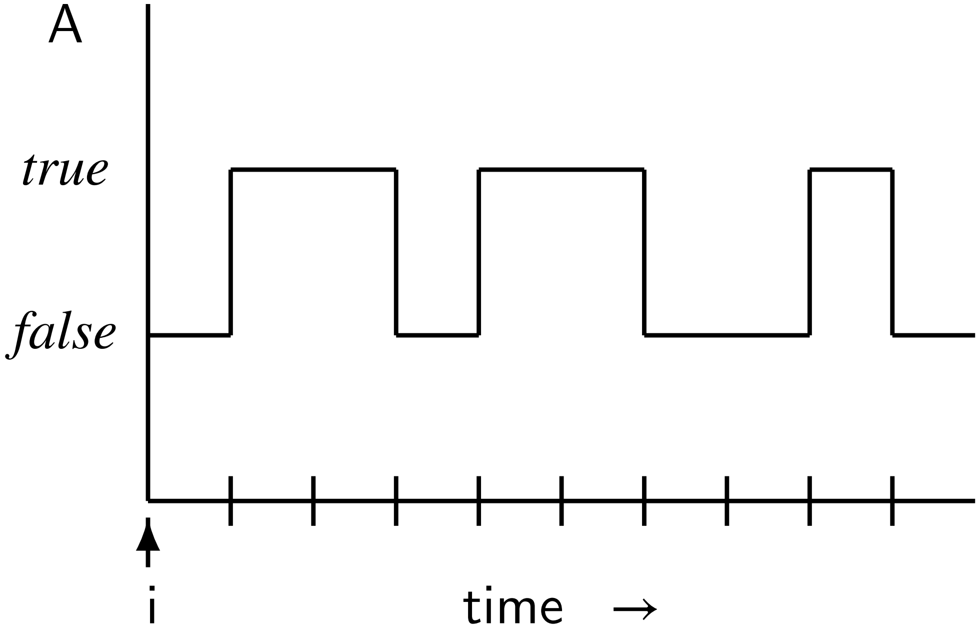
(159) **Distributivity of $\Box\Diamond$ over $\wedge$:** $\Box\Diamond(p \wedge q) \Rightarrow \Box\Diamond p \wedge \Box\Diamond q$

(160) **Distributivity of $\Diamond\Box$ over $\vee$:** $\Diamond\Box p \vee \Diamond\Box q \Rightarrow \Diamond\Box(p \vee q)$

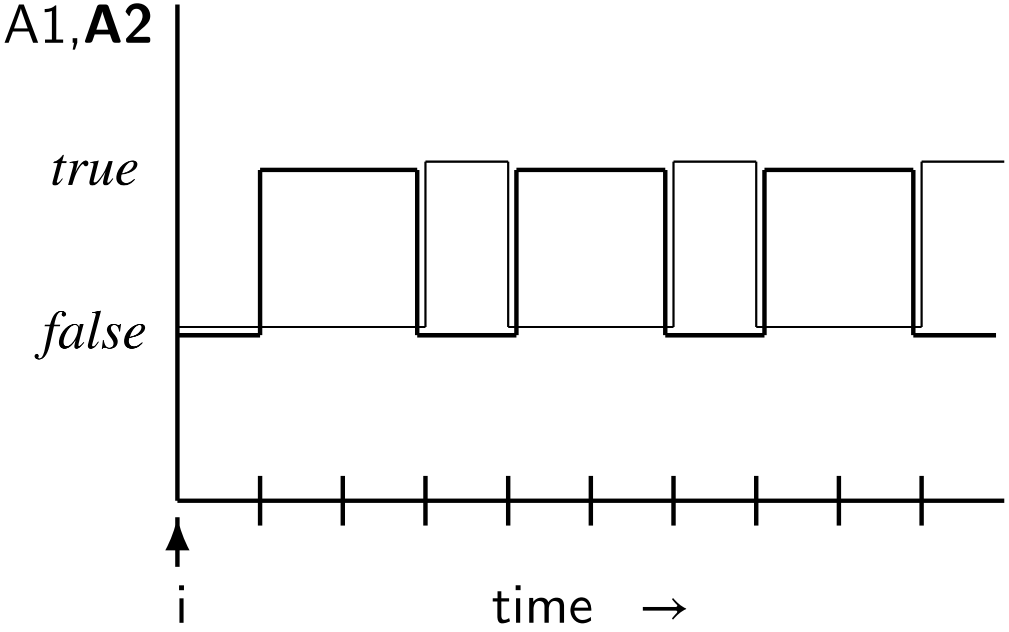
(161) **Distributivity of $\Box\Diamond$ over $\vee$:** $\Box\Diamond(p \vee q) \equiv \Box\Diamond p \vee \Box\Diamond q$

(162) **Distributivity of $\Diamond\Box$ over $\wedge$:** $\Diamond\Box(p \wedge q) \equiv \Diamond\Box p \wedge \Diamond\Box q$

$\Box\Diamond A$

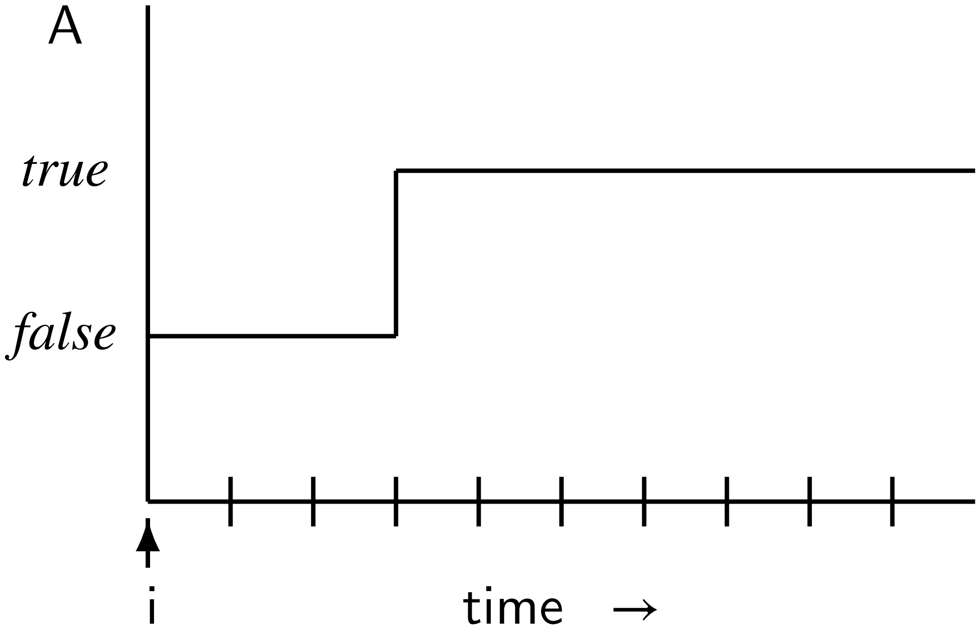


$\Box\Diamond A1 \wedge \Box\Diamond A2$

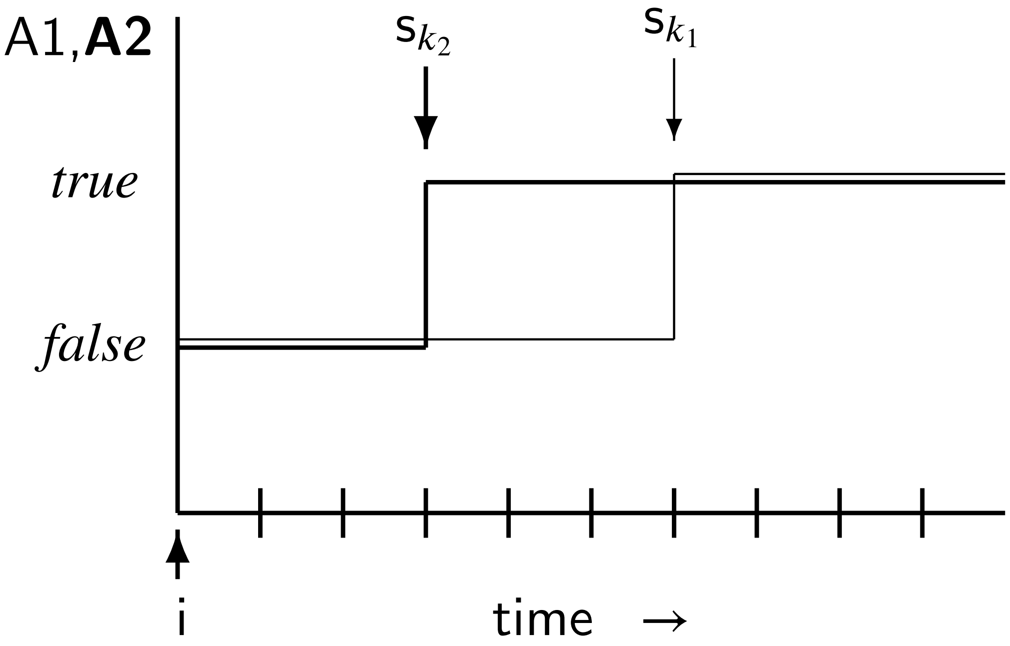


(159) **Distributivity of $\Box\,\Diamond$ over $\wedge$:** $\Box\,\Diamond\,(p \wedge q) \Rightarrow \Box\,\Diamond\,p \wedge \Box\,\Diamond\,q$

$$\Diamond\Box A$$



$\Diamond\Box A1 \wedge \Diamond\Box A2$



(162) **Distributivity of $\Diamond\,\Box$ over $\wedge$:** $\Diamond\,\Box\,(p \wedge q) \equiv \Diamond\,\Box\,p \wedge \Diamond\,\Box\,q$

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(168) **Progress proof rule:** $\Diamond \Box p \wedge \Box (\Box p \Rightarrow \Diamond q) \Rightarrow \Diamond q$