Iteration and Invariants

Tail recursion:

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- A good optimizing compiler can convert a tail recursive program into an iterative one (as a loop).

Not tail recursion:

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Because after the recursive call additional processing must be done before the value is returned

General idea:

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- The helper function is tail recursive, and calls itself.
- The helper function has an extra parameter.
- The extra processing is done in the parameters of the helper function.

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Helper function, with two parameters a and b that computes a * b!

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$$a \cdot b! = (a \cdot b) \cdot (b - 1)!$$

Not tail recursion:

$$n! = \underset{\uparrow}{\underline{n \ast}}\, (n-1)!$$

Helper function, with two parameters a and b that computes a * b!

$$a \cdot b! = (a \cdot b) \cdot (b-1)!$$

The main function calls the helper function with a value of one for *a* and *n* for *b*.

Write `factorial` and `factorial-product`.

Prove `factorial-product` is correct.

```
(define factorial-product
  (lambda (a b)  ; Returns a*b!, b >= 0.
    (if (= b 0)
        a
        (factorial-product (* a b) (- b 1)))))
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Base case

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Code inspection: `(factorial-product a 0)` returns a.

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Math: $a \cdot 0! = a$

Therefore, correct in base case.

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Inductive case

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<u>Inductive case</u>

Prove that

`(factorial-product x b)` terminates

with value $x \cdot b!$

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Inductive case

Prove that

`(factorial-product x b)` terminates

with value $x \cdot b!$

assuming that

`(factorial-product y (- b 1))` terminates

with value $y \cdot (b - 1)!$

as the inductive hypothesis.

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Value returned by `(factorial-product a b)`

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$= \quad \langle \text{Math} \rangle$

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$a \cdot b!$ $\blacksquare$

The `power` function

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```
> (power 4 5)
1024
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`(power 4 5)` returns 4 to the power 5.

Not tail recursion:

$$b^e = \underset{\uparrow}{\underline{b \cdot}}\, b^{e-1}$$

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Tail recursion:

$$(a) \cdot b^e = (ab) \cdot b^{e-1}$$

Write `power` and `power-product`.

Exponentiation is not associative

$$(3^4)^5 \neq 3^{(4^5)}$$

Exponentiation is not associative

$$(3^4)^5 \neq 3^{(4^5)}$$

$$3^{20} \neq 3^{1024}$$

Fermat numbers

$$F_n = 2^{(2^n)} + 1$$

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The first few Fermat numbers

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Fermat numbers

$$F_n = 2^{(2^n)} + 1$$

The first few Fermat numbers

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$$n = 1 \Rightarrow 2^2 + 1 = 5$$

$$n = 2 \Rightarrow (2^2)^2 + 1 = 17$$

$$n = 3 \Rightarrow ((2^2)^2)^2 + 1 = 257$$

Fermat numbers

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2 repeatedly squared 3 times

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`(repeatedly-square 2 1)` should return $2^2 = 4$

Fermat numbers

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`(repeatedly-square 2 0)` should return 2

`(repeatedly-square 2 1)` should return $2^2 = 4$

`(repeatedly-square 2 2)` should return $(2^2)^2 = 16$

Fermat numbers

$$b^{(2^n)}$$

Fermat numbers

$$\begin{aligned} & b^{(2^n)} \\ = \quad & \langle \text{Math}, 2^n = 2 \cdot 2^{n-1} \rangle \end{aligned}$$

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 & (b^2)^{2^{n-1}}
\end{array}
$$

b repeatedly squared n times equals b^2 repeatedly squared $n-1$ times.

Write `fermat-number`
and `repeatedly-square`.

Perfect number

- The sum of its divisors is twice the number, or
- The number is equal to the sum of its divisors other than itself.

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- The number is equal to the sum of its divisors other than itself.

$$\left.\begin{aligned} 2\cdot 6 &= 1+2+3+6 \\ 6 &= 1+2+3 \end{aligned}\right\} \Rightarrow 6 \text{ is perfect.}$$

There are no known odd perfect numbers.

Suppose you have a function `sum-of-divisors` such that

`(sum-of-divisors 4)` returns 1+2+4,

`(sum-of-divisors 5)` returns 1+5,

`(sum-of-divisors 6)` returns 1+2+3+6, etc.

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Write `perfect?` such that

`(perfect? 4)` returns `#f`,

`(perfect? 5)` returns `#f`,

`(perfect? 6)` returns `#t`, etc.

The shape of `sum-of-divisors`

```
(define sum-of-divisors
  (lambda (n)









    (sum-from-plus 1 0)))
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`sum-from-plus` is defined
inside `sum-of-divisors`.
So, it has access to `n`.

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```
    (sum-from-plus 1 0)))
```

low addend

The shape of `sum-of-divisors`

```
(define sum-of-divisors
  (lambda (n)
```

`sum-from-plus` returns the sum of all the divisors of `n` that are greater than or equal to `low` plus the `addend`.

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1 2 3 4 5 6 7 8 9 10 11 12

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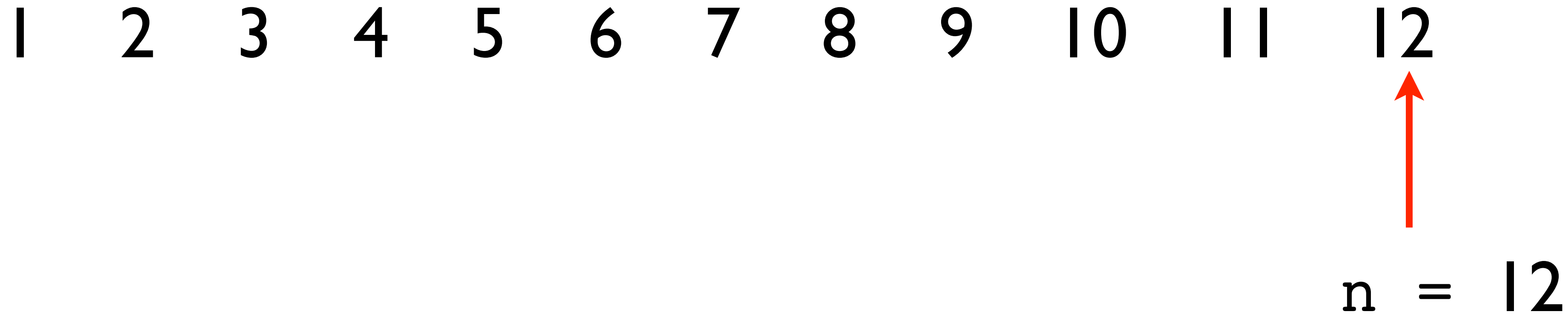

1 2 3 4 5 6 7 8 9 10 11 12

`n = 12`

`sum-from-plus` returns the sum of all the divisors of `n` that are greater than or equal to `low` plus the `addend`.

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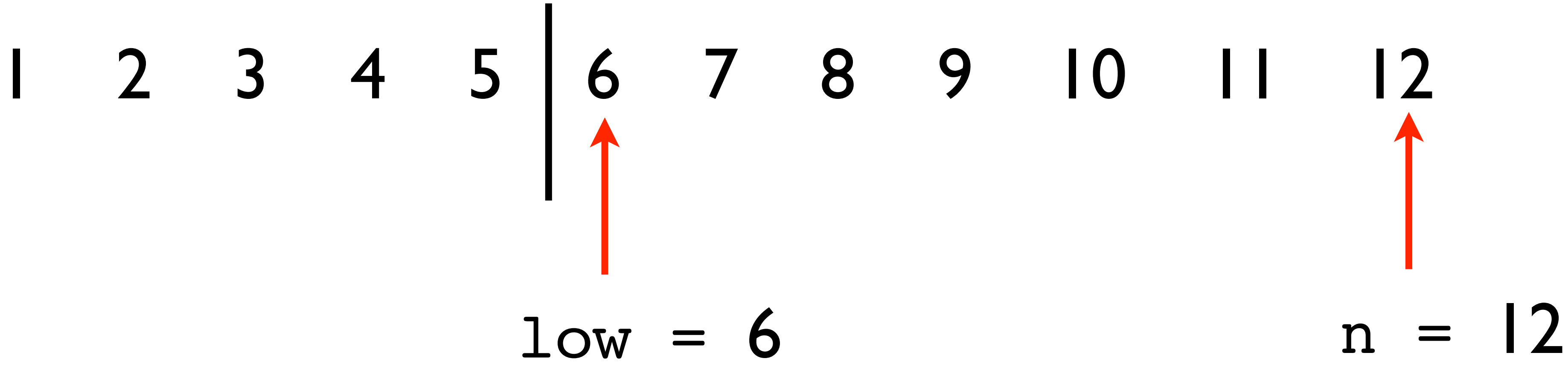

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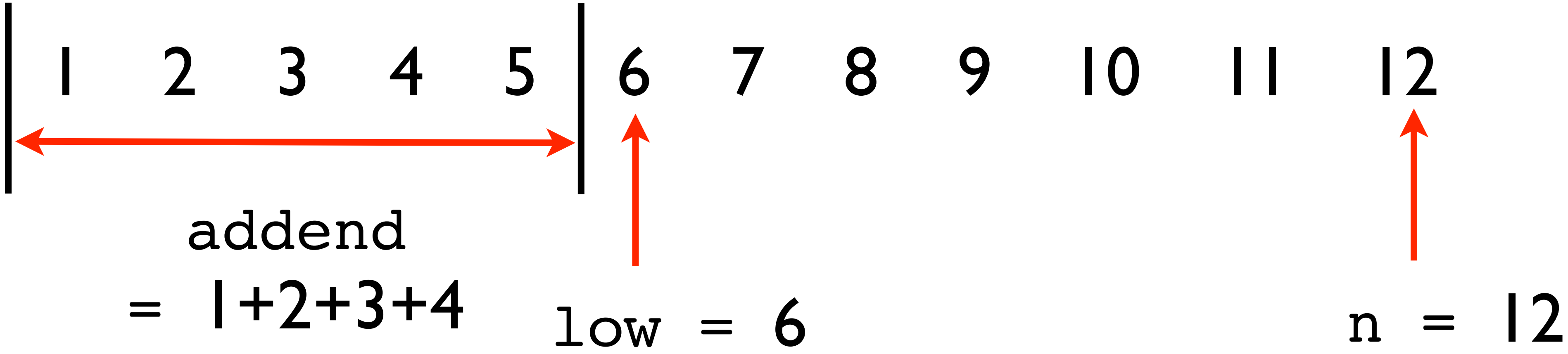

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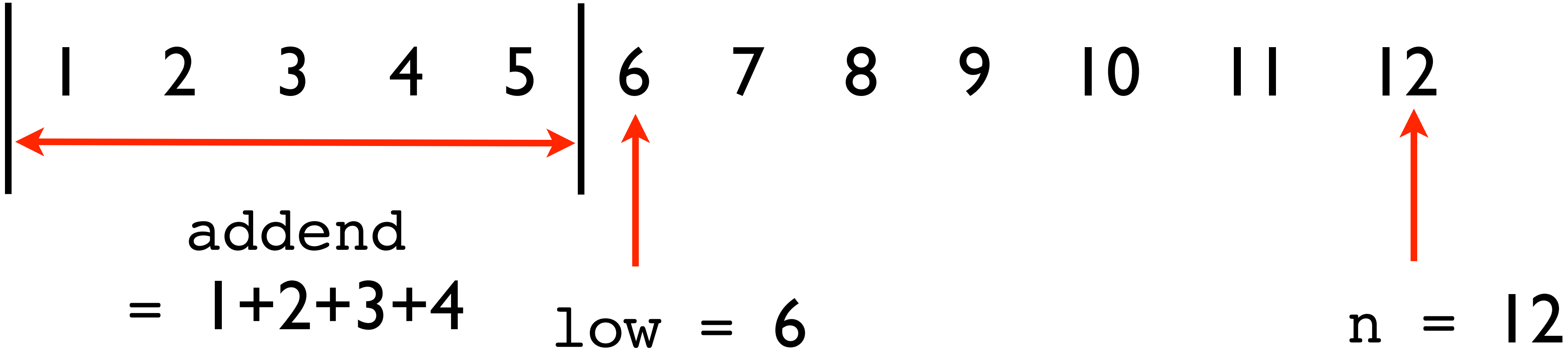

1 2 3 4 5 | 6 7 8 9 10 11 12

`addend`
= 1+2+3+4

`low = 6`

`n = 12`

`(sum-from-plus 6 10)`

sum-of-divisors

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(define sum-of-divisors
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      (sum-from-plus 1 0)))
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addend

sum-of-divisors

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(define sum-of-divisors
  (lambda (n)
    (define sum-from-plus ; sum of all divisors of n which are
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                           (if (divides? low n)




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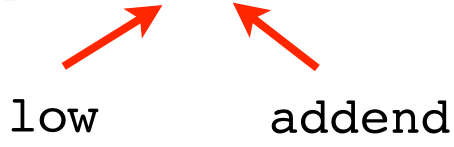

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low

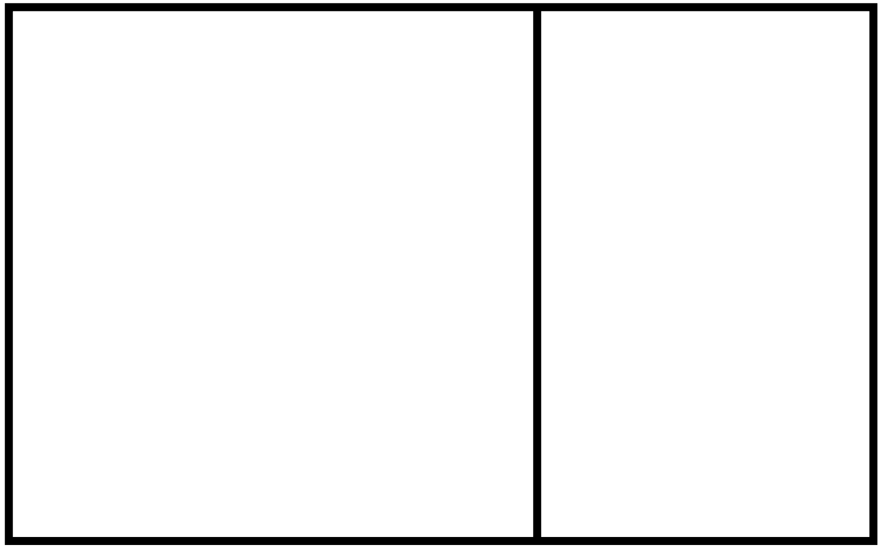
addend

Hallmarks of pure functional programming

- A function returns a value.
- There is no call by reference.
- All parameters are called by value.
- There are no loops.
- Repetition is achieved by recursion.
- There is no assignment statement.

The Golden Ratio

The Golden Ratio



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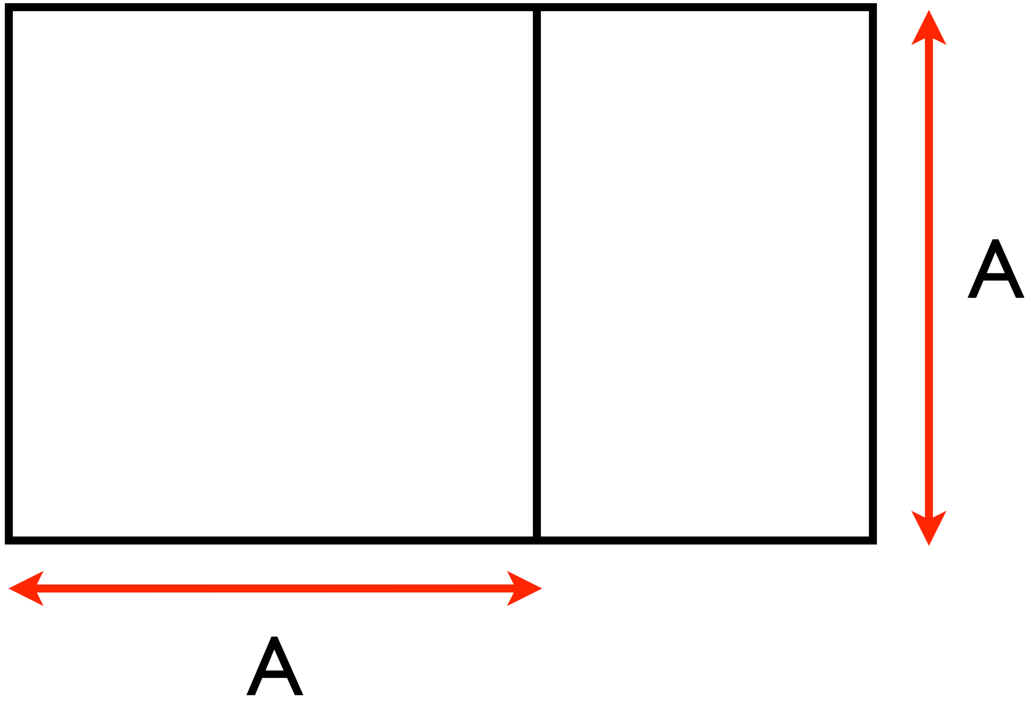

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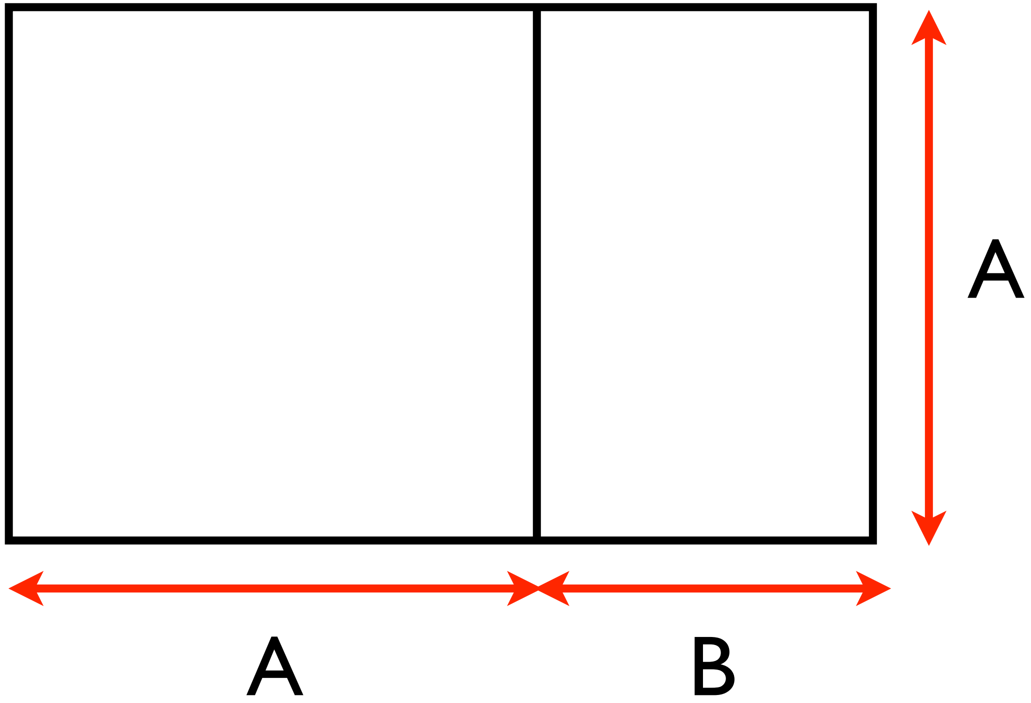

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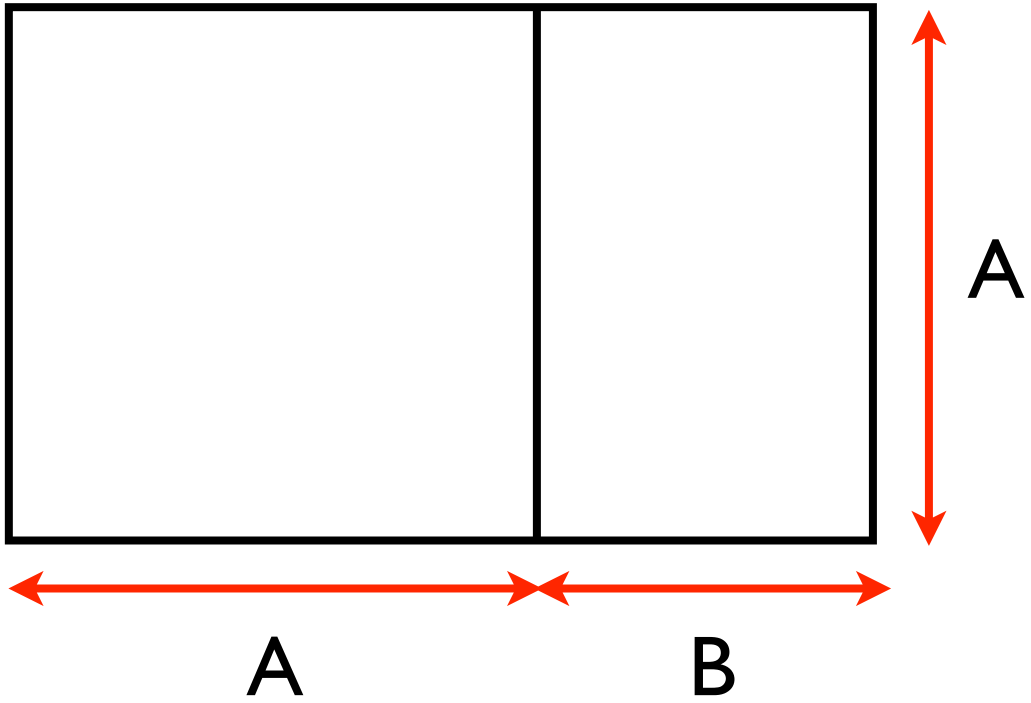
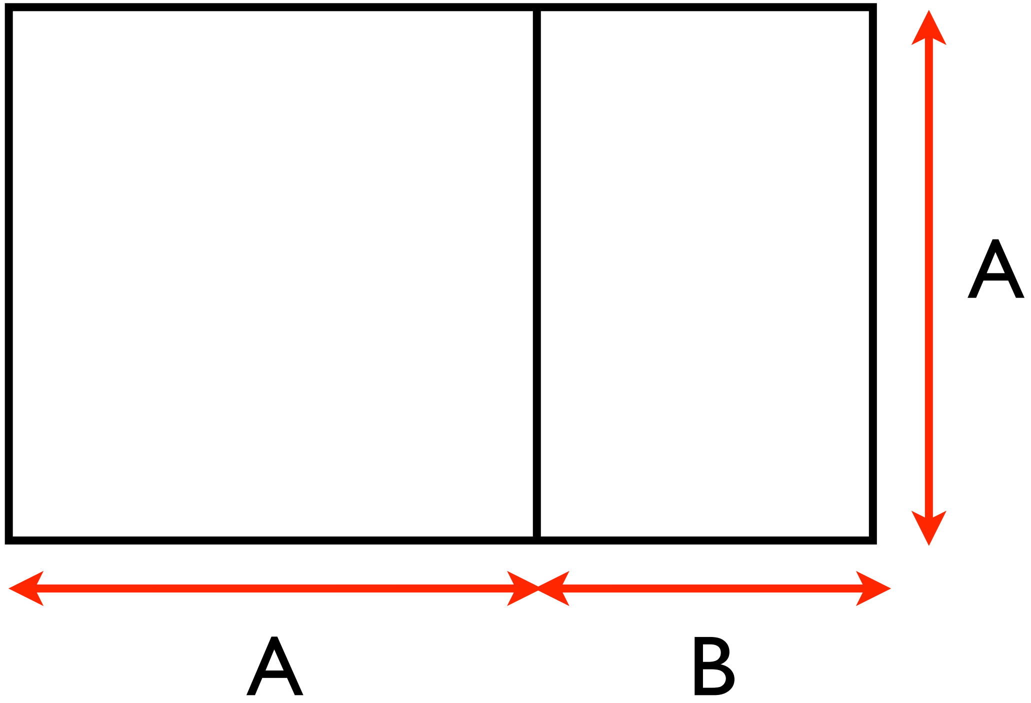


$$\frac{A}{B} = \frac{A + B}{A}$$

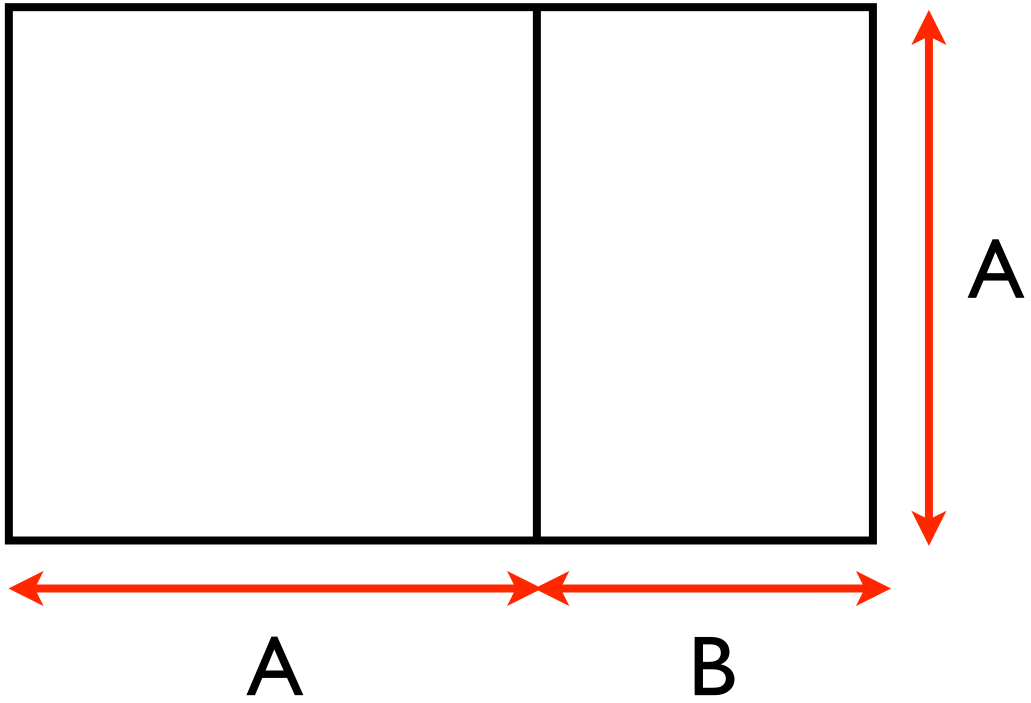
The Golden Ratio



$$\frac{A}{B} = \frac{A+B}{A}$$

$$= 1 + \frac{B}{A}$$

The Golden Ratio



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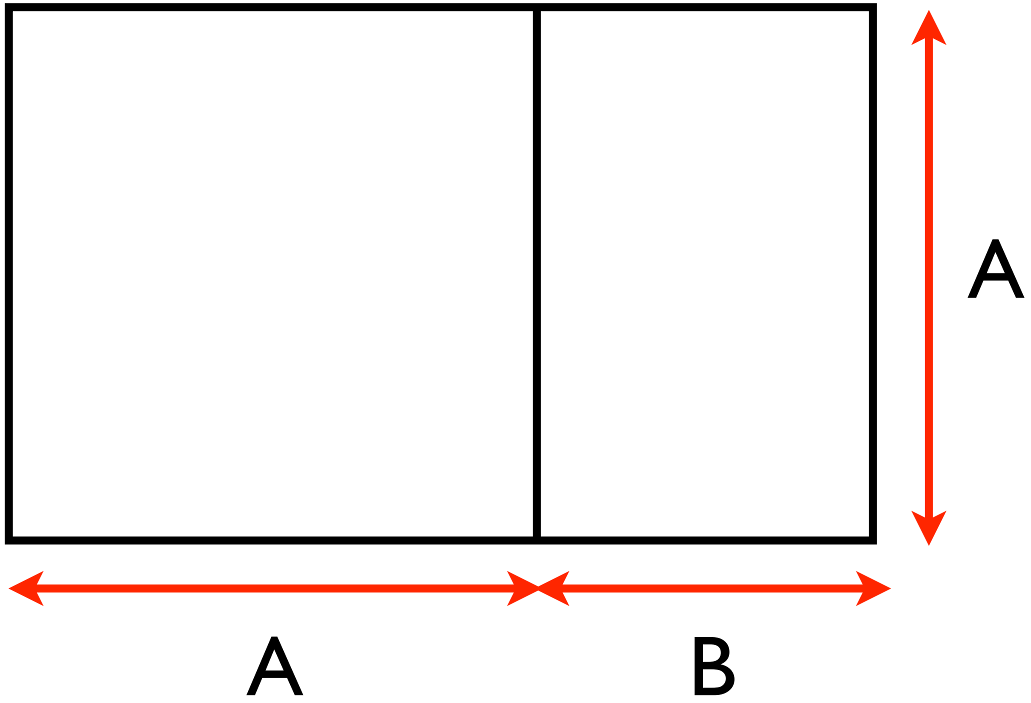


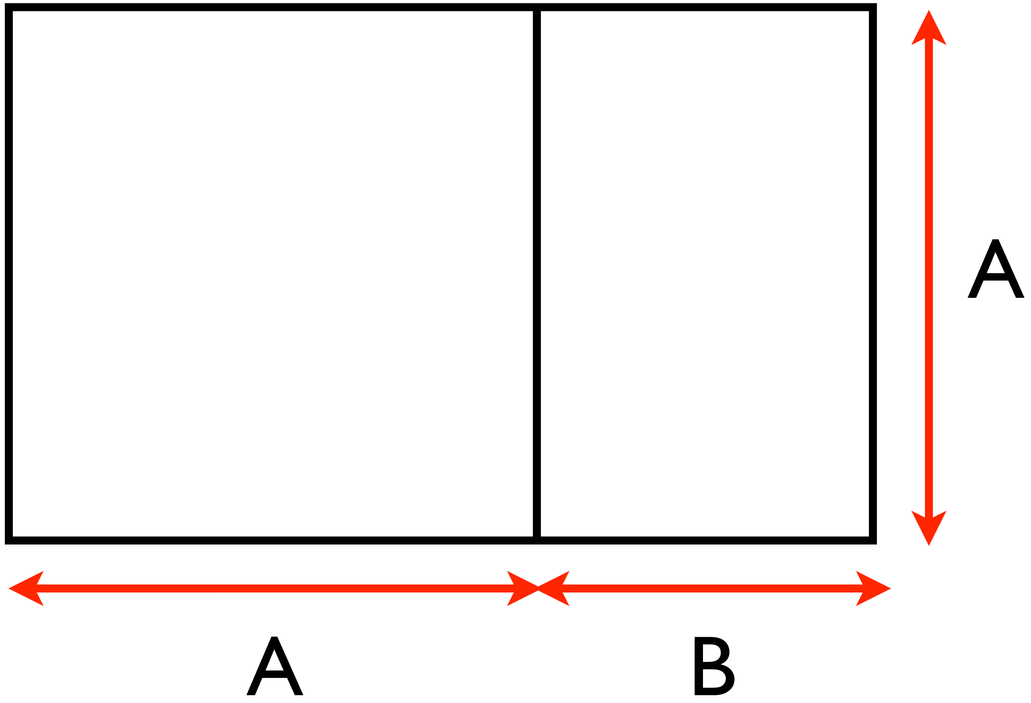
$$\frac{A}{B} = \frac{A+B}{A}$$

$$= 1 + \frac{B}{A}$$

$$= 1 + \frac{1}{A/B}$$

$$\phi = A/B$$

The Golden Ratio



$$\frac{A}{B} = \frac{A+B}{A}$$

$$= 1 + \frac{B}{A}$$

$$= 1 + \frac{1}{A/B}$$

$$\phi = A/B$$

$$\phi = 1 + \frac{1}{\phi}$$

Successive approximations of the Golden Ratio

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$$\phi_0 = 1$$

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$$\phi_1 = 1 + \frac{1}{\phi_0} = 2$$

$$\phi_2 = 1 + \frac{1}{\phi_1} = \frac{3}{2}$$

Successive approximations of the Golden Ratio

$$\begin{aligned}
\phi_0 &= 1 \\
\phi_1 &= 1 + \frac{1}{\phi_0} = 2 \\
\phi_2 &= 1 + \frac{1}{\phi_1} = \frac{3}{2} \\
\phi_3 &= 1 + \frac{1}{\phi_2} = \frac{5}{3}
\end{aligned}$$

Successive approximations of the Golden Ratio

$$\phi_0 = 1$$

$$\phi_1 = 1 + \frac{1}{\phi_0} = 2$$

$$\phi_2 = 1 + \frac{1}{\phi_1} = \frac{3}{2}$$

$$\phi_3 = 1 + \frac{1}{\phi_2} = \frac{5}{3}$$

$$\phi_4 = 1 + \frac{1}{\phi_3} = \frac{8}{5}$$

Successive approximations of the Golden Ratio

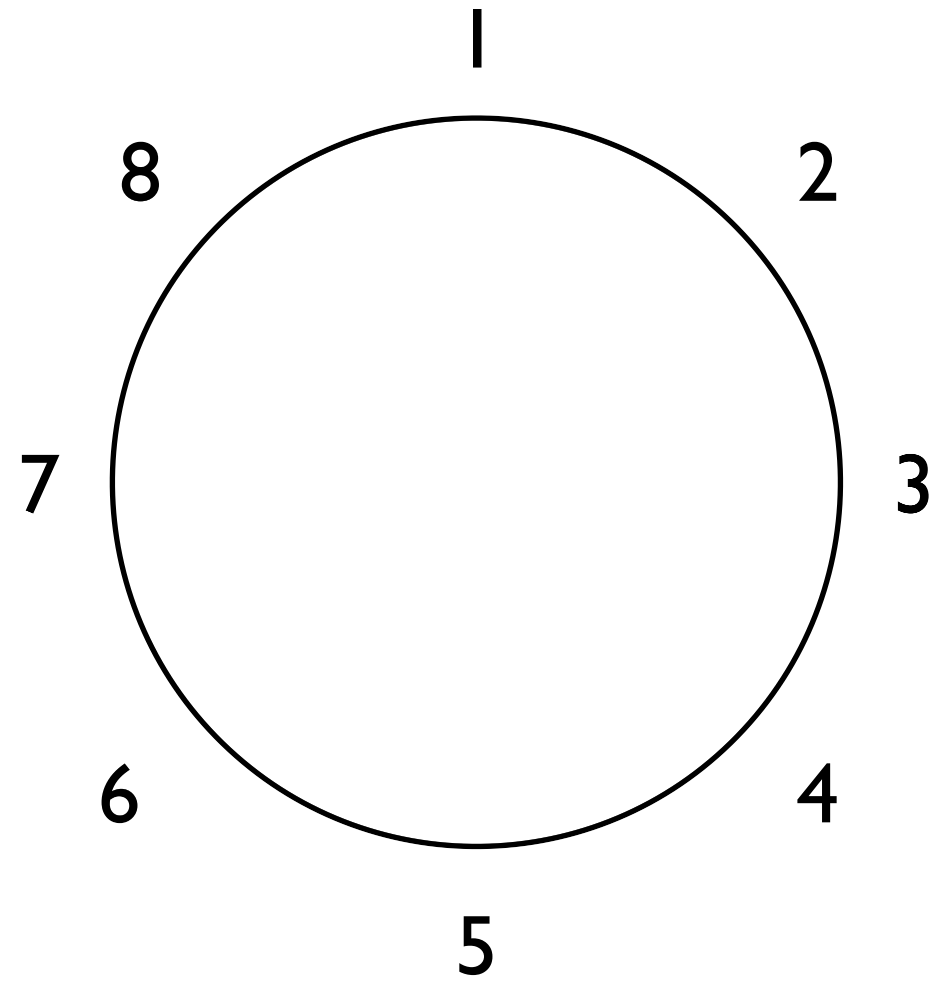
`(phi 0)` returns 1,
`(phi 1)` returns 2,
`(phi 2)` returns 3/2,
`(phi 3)` returns 5/3,
`(phi 4)` returns 8/5, etc.

Write `phi`

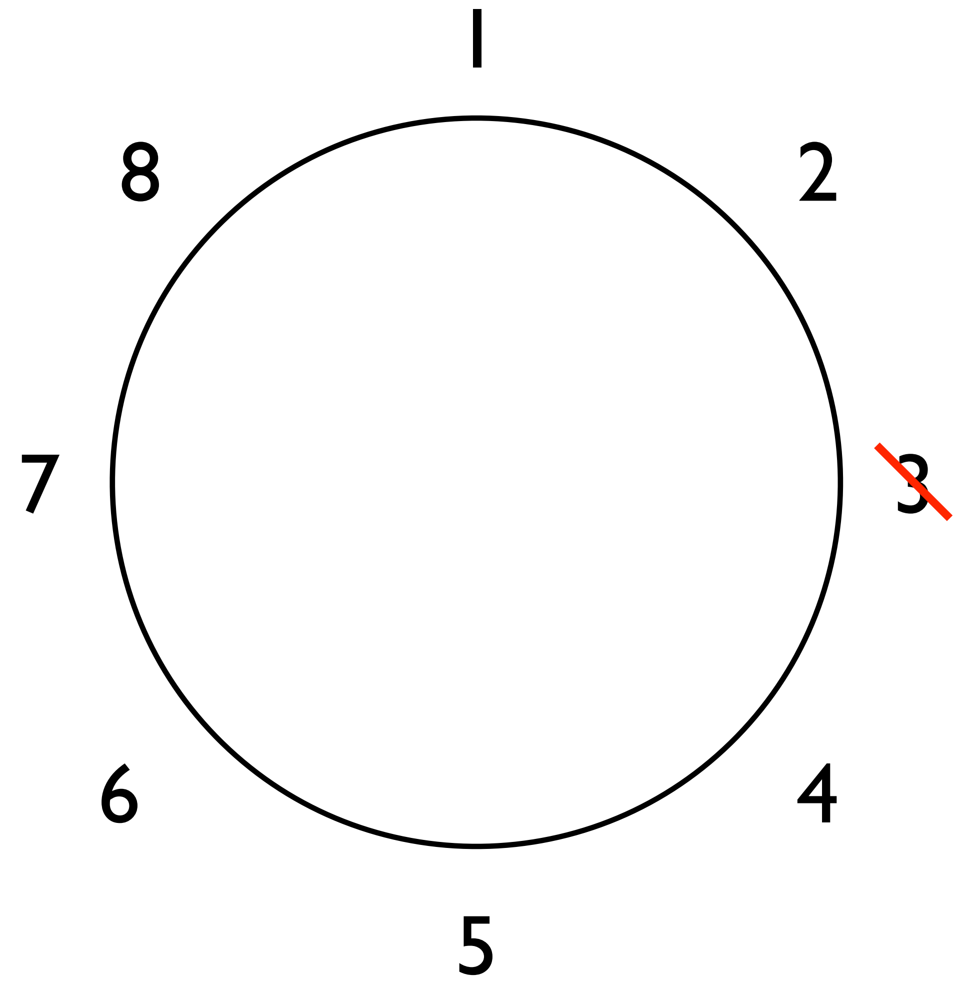
The Josephus Problem

Rule:
Everybody stands in a circle.
Starting with the first, kill every third person.
Which two people remain alive?

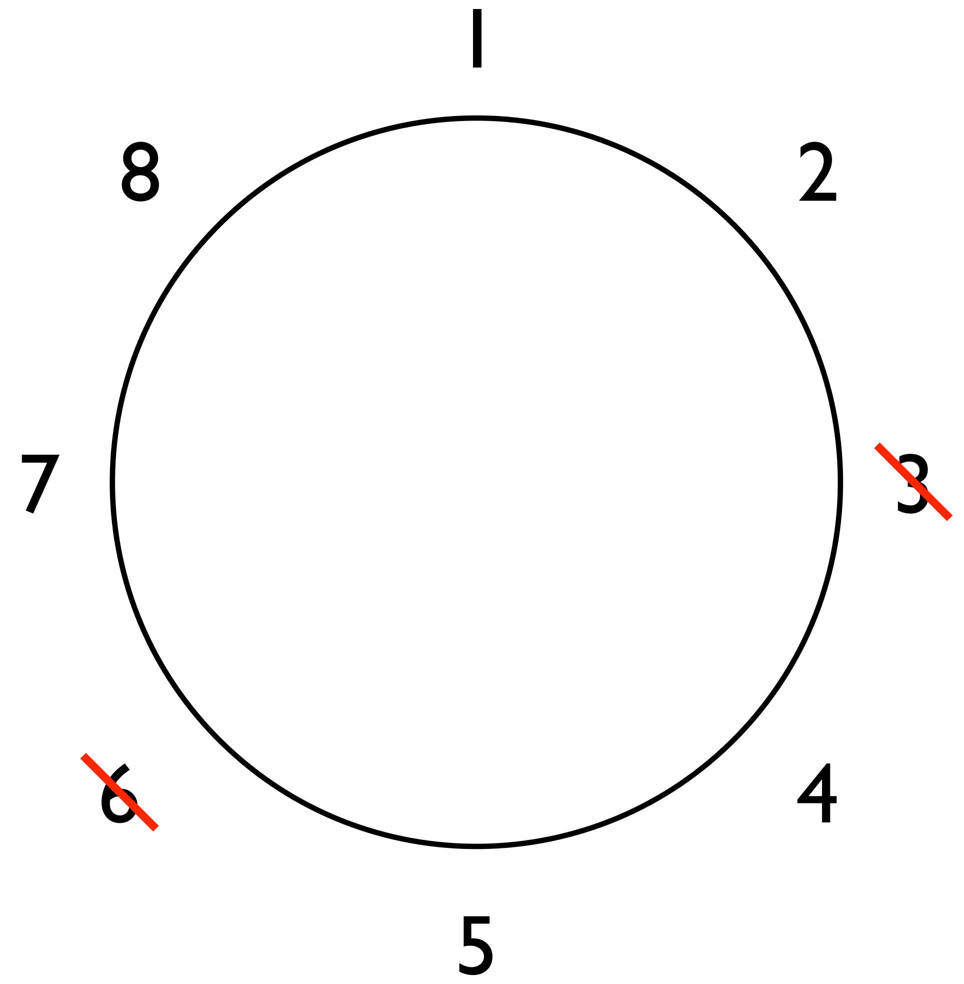
The Josephus Problem



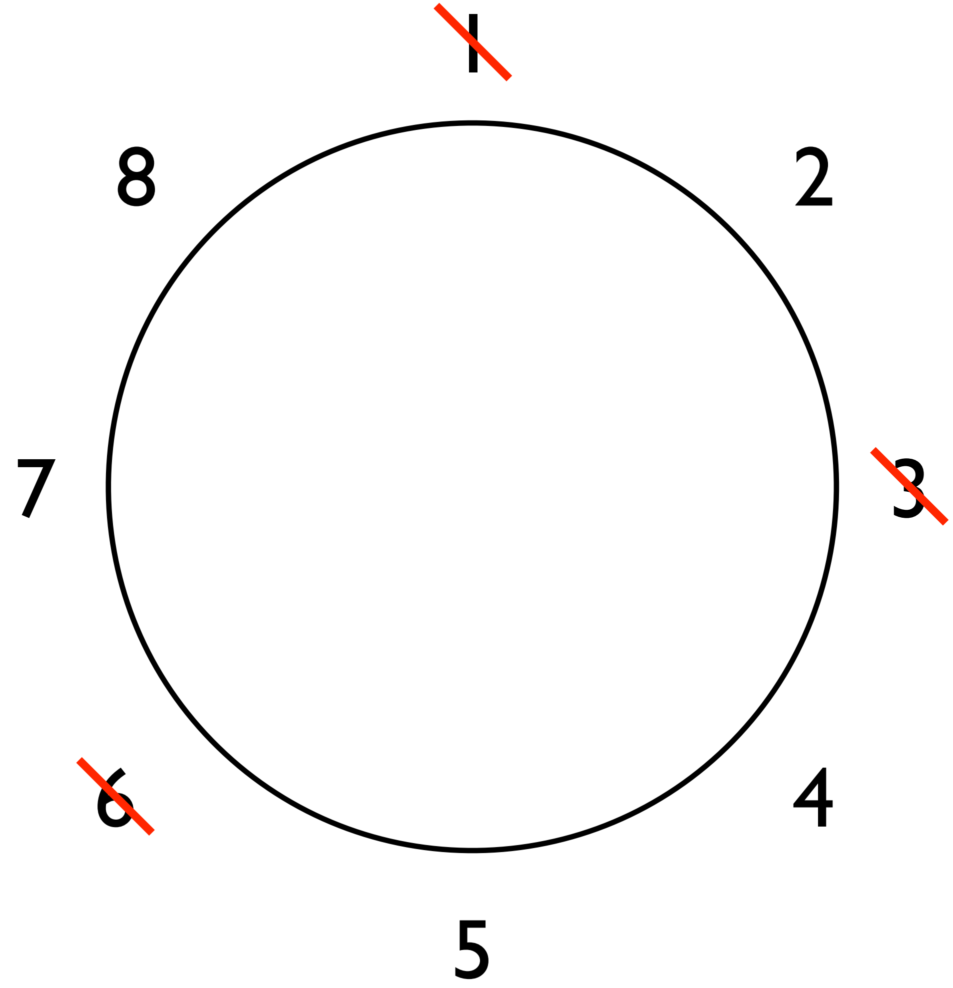
The Josephus Problem



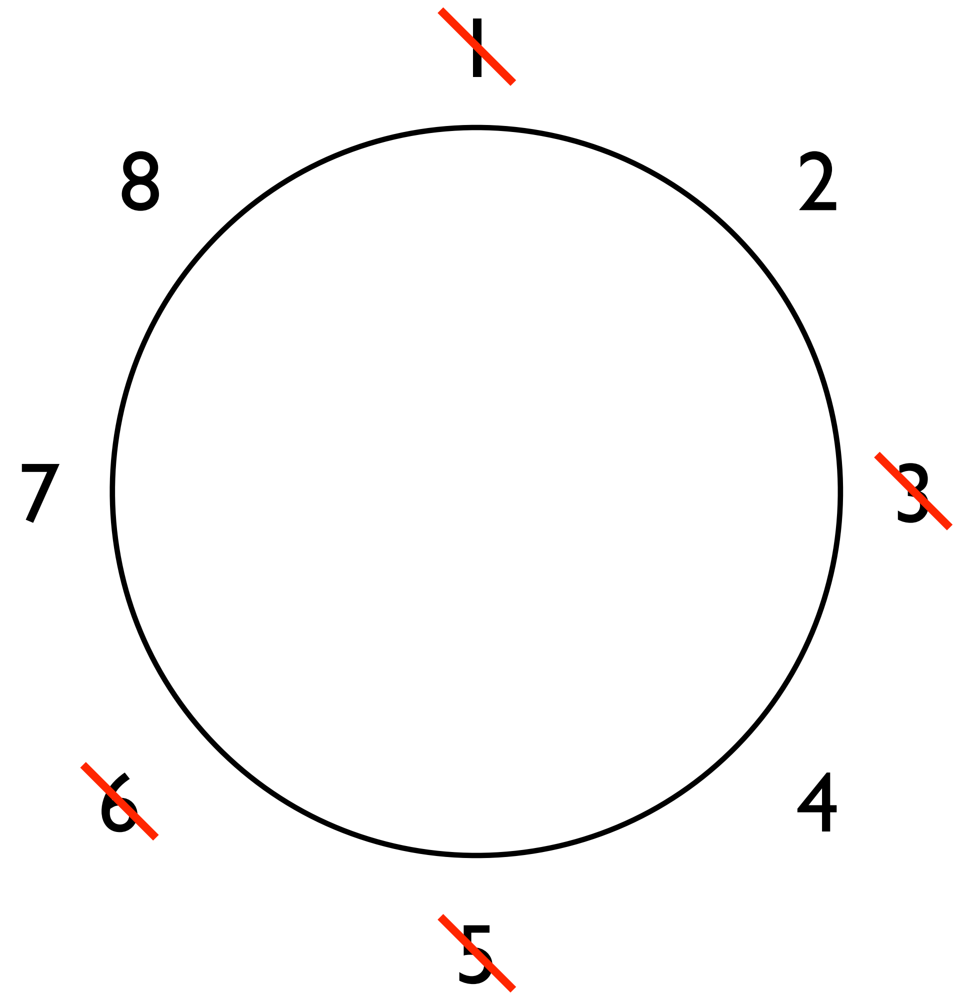
The Josephus Problem



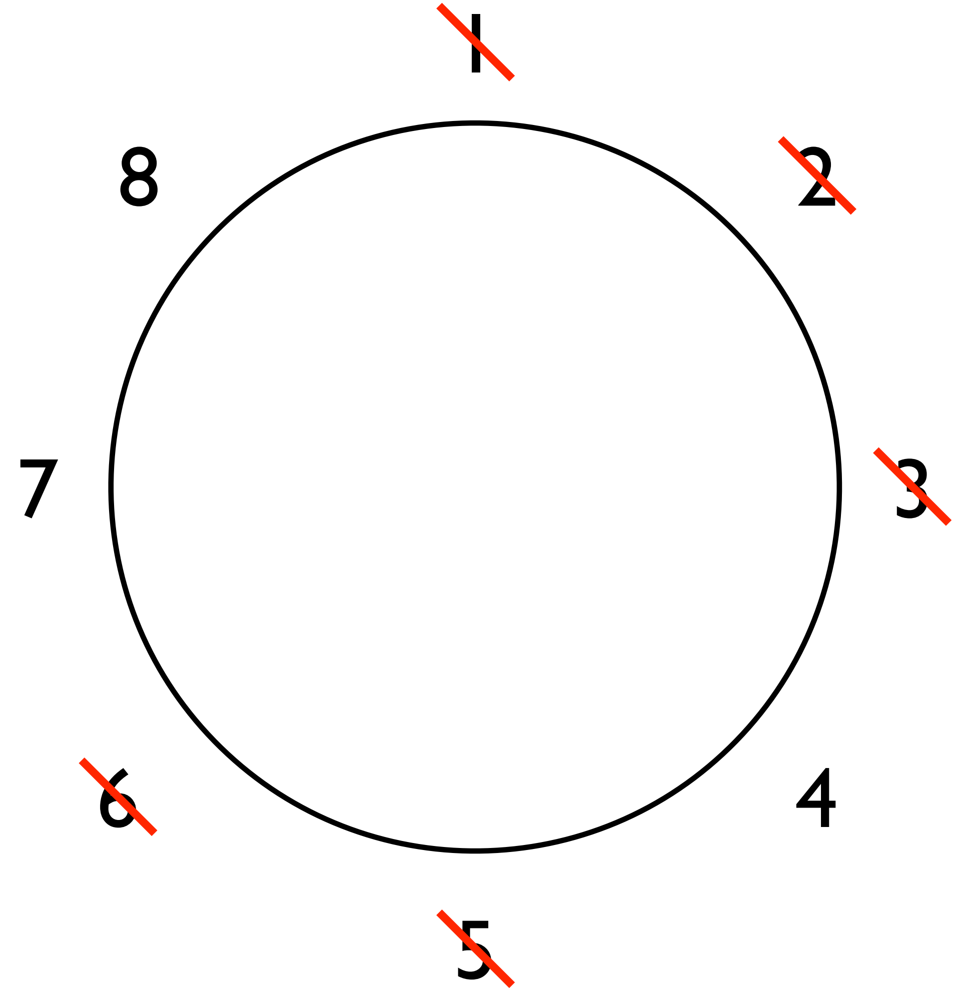
The Josephus Problem



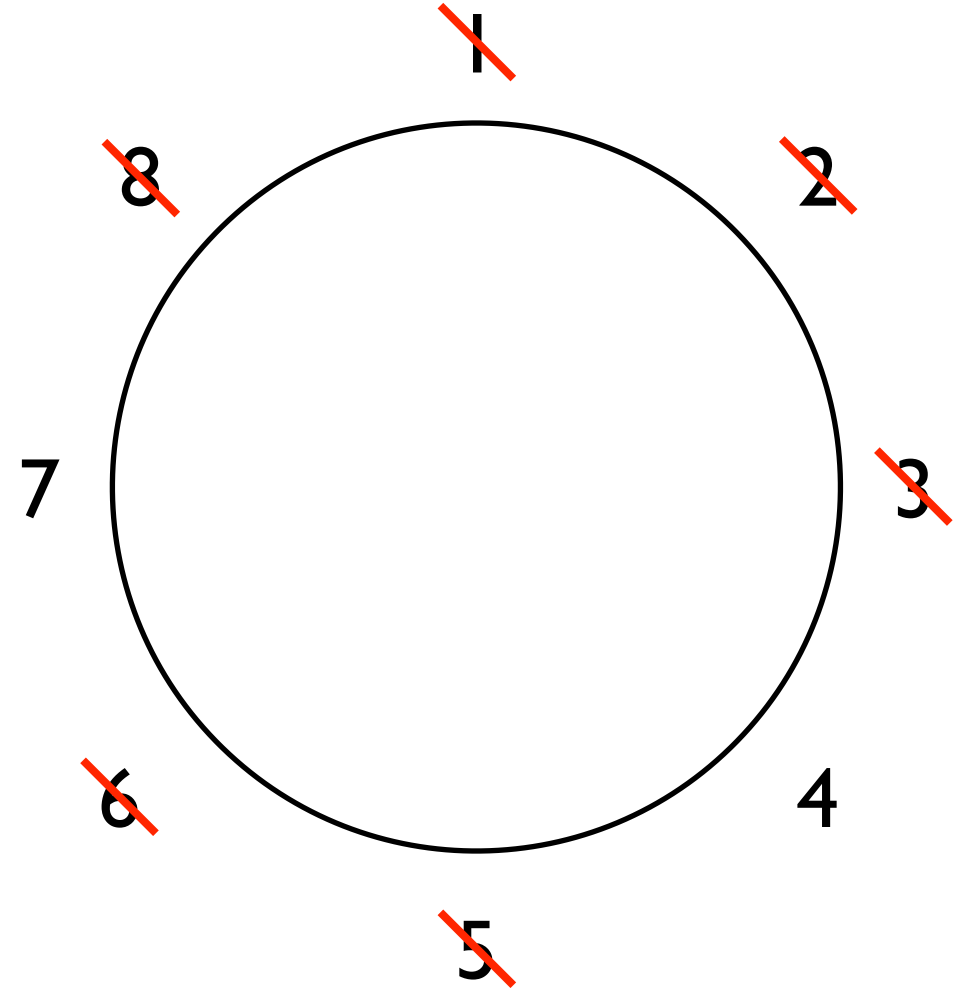
The Josephus Problem



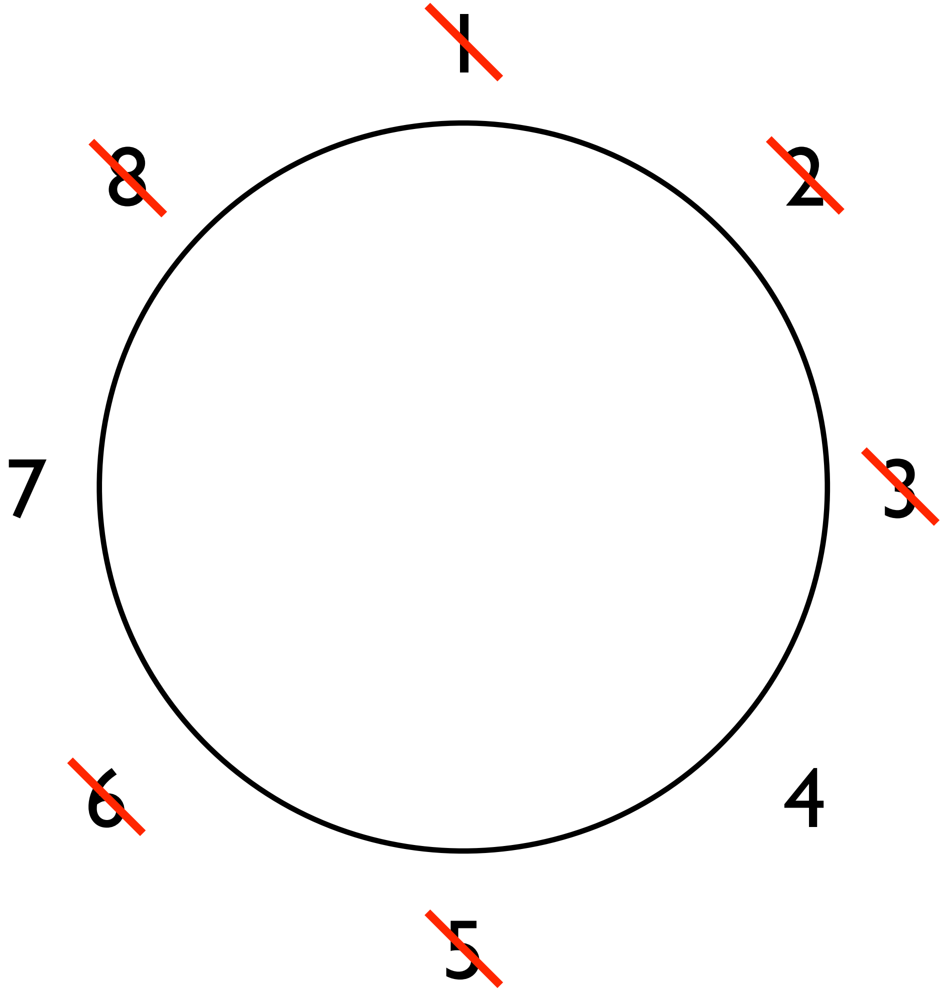
The Josephus Problem



The Josephus Problem

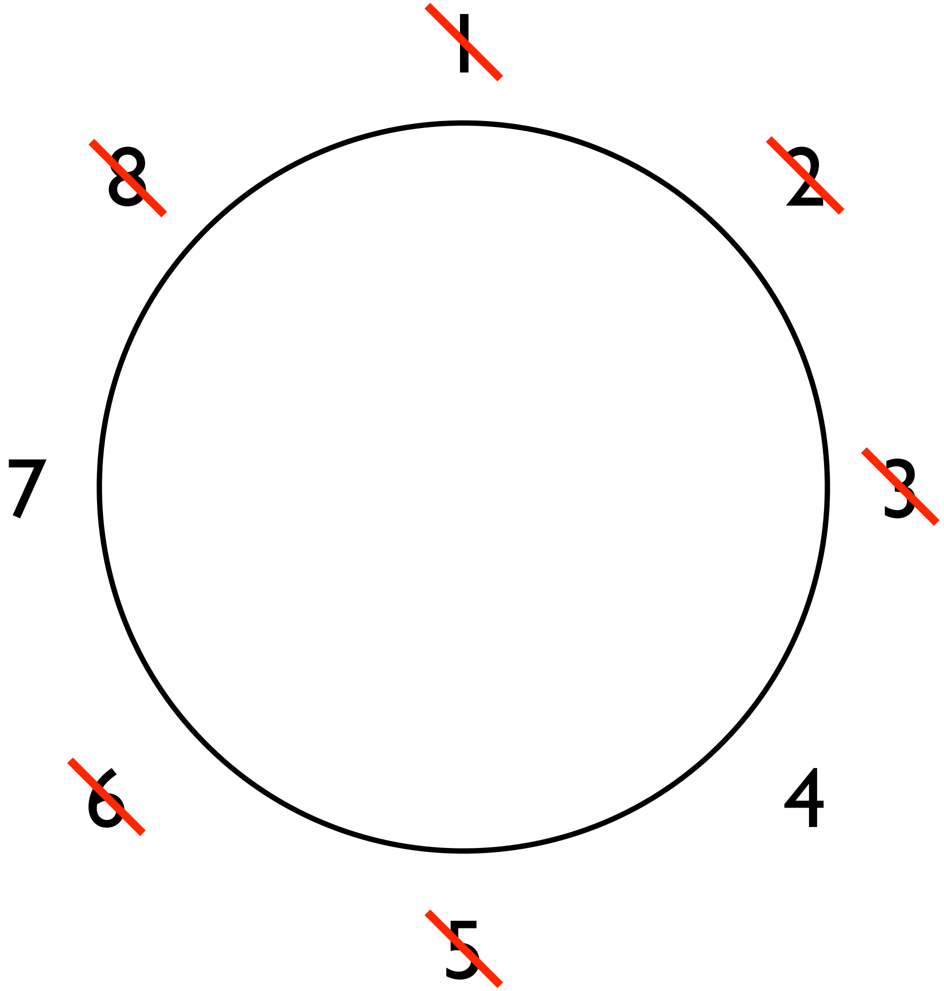


The Josephus Problem



`(survives? 4 8)` returns `#t`

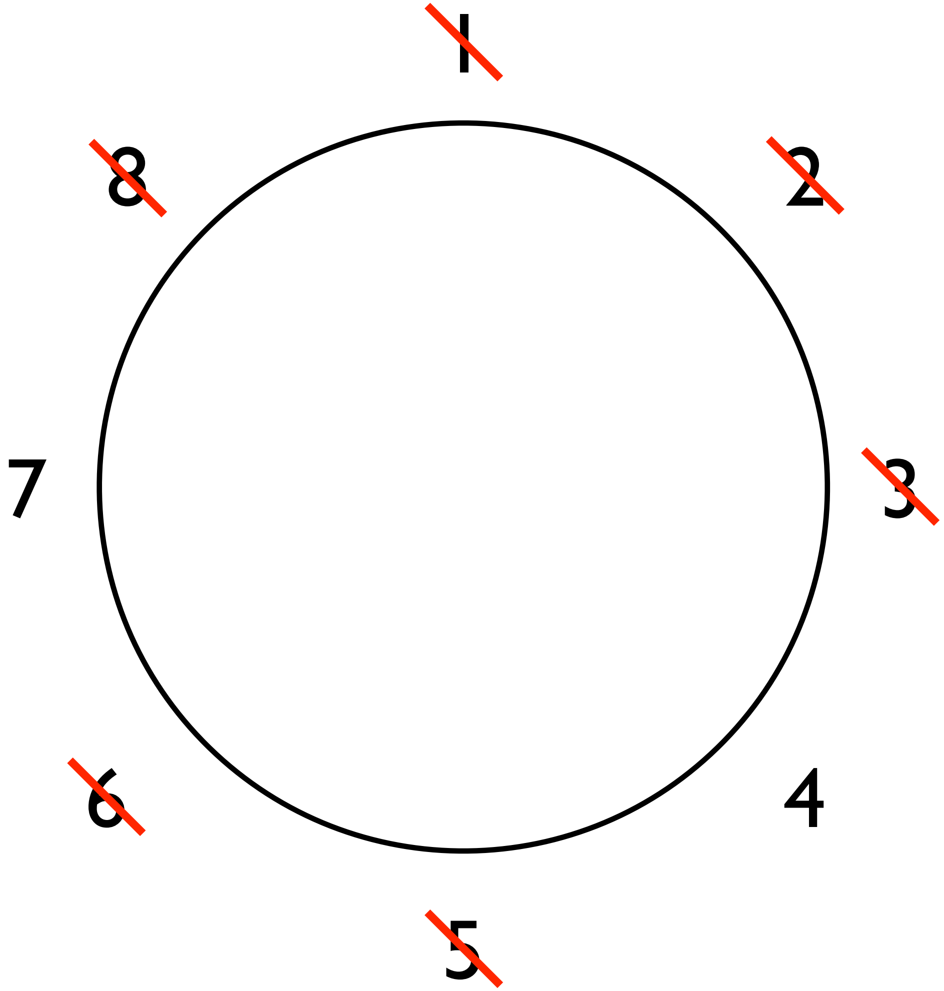
The Josephus Problem



`(survives? 4 8)` returns `#t`

`(survives? 7 8)` returns `#t`

The Josephus Problem



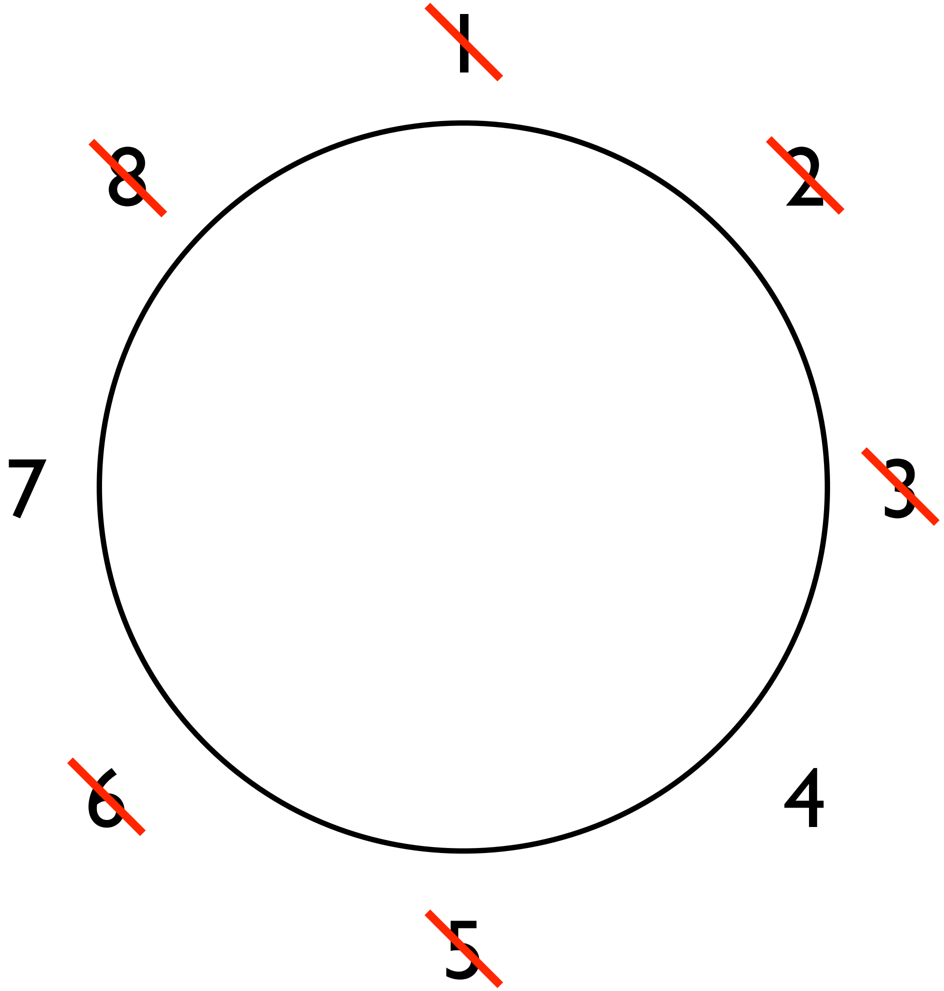
`(survives? 4 8)` returns `#t`

`(survives? 7 8)` returns `#t`

`(survives? k 8)` returns `#f`

for all other values of `k`

The Josephus Problem



`(survives? 4 8)` returns `#t`

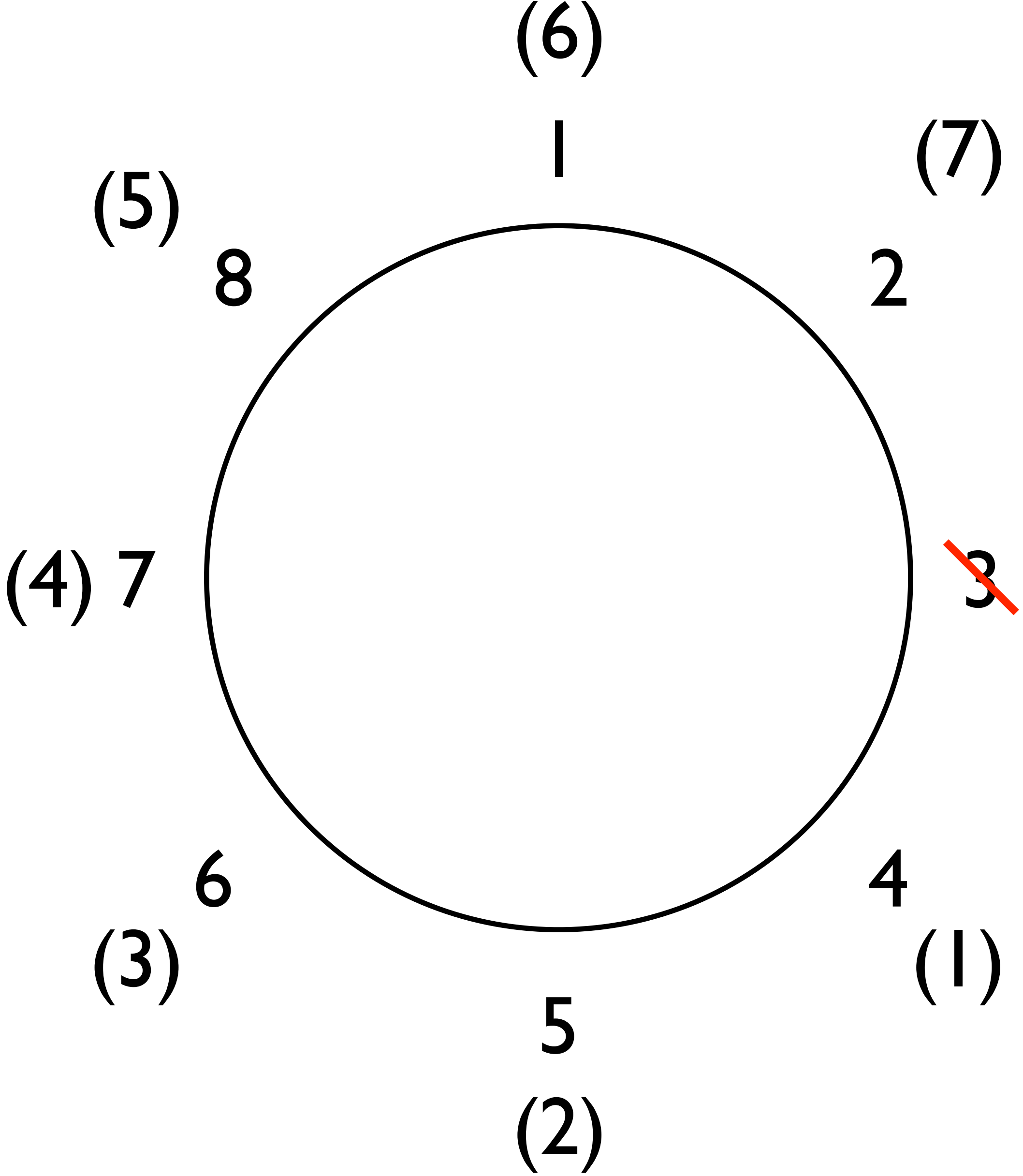
`(survives? 7 8)` returns `#t`

`(survives? k 8)` returns `#f`

for all other values of `k`

Exercise for the student:
Write `survives?`

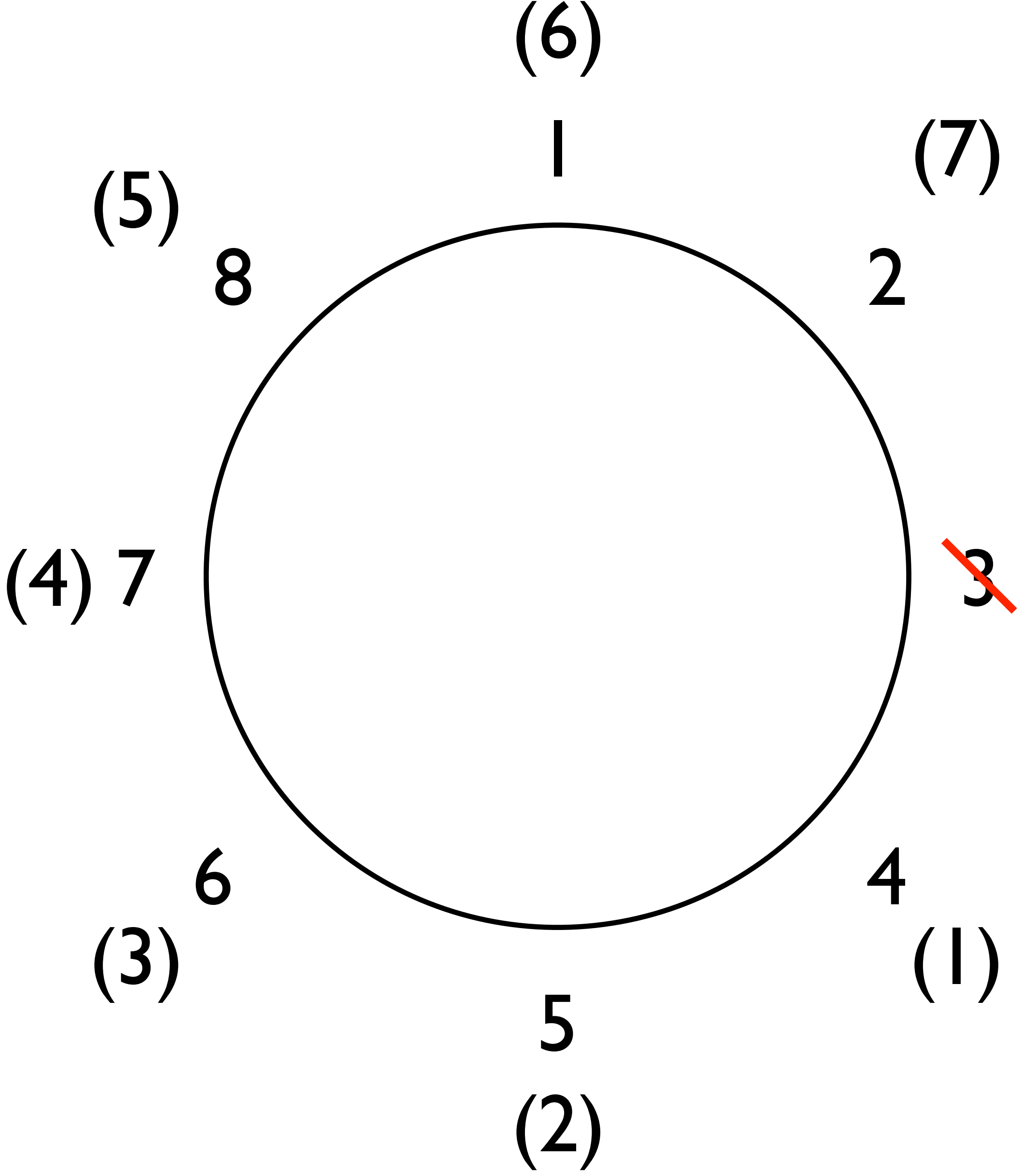
The Josephus Problem



Hint:

Parentheses are the new numbers after the first person is killed.

The Josephus Problem



Hint:

Parentheses are the new numbers after the first person is killed.

`(survives? 6 8)`

is recursively the same as

`(survives? 3 7)`