Higher-Order Procedures
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In the same way you can pass data values as parameters in a function, you can pass function as a parameter in another function.
(define power
  (lambda (b e)
    (if (= e 1)
      b
      (* (power b (- e 1)) b))))

(define stack-copies-of
  (lambda (quantity image)
    (if (= quantity 1)
      image
      image
      (stack (stack-copies-of (- quantity 1) image) image))))
How are the functions similar?

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How are the functions different?

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  (lambda (quantity image)
    (if (= quantity 1)
      image
      (stack (stack-copies-of (- quantity 1) image) image))))

The form is the same. The functions are different.
(define together-copies-of
  (lambda (combine quantity thing)
    (if (= quantity 1)
        thing
        (combine (together-copies-of combine
                    (- quantity 1)
                    thing)
                  thing))))
(define together-copies-of
  (lambda (combine quantity thing)
    (if (= quantity 1)
        thing
        (combine (together-copies-of combine
                      (- quantity 1)
                      thing)
                  thing))))

The first parameter in the function `together-copies-of` is a function.
(define together-copies-of
    (lambda (combine quantity thing)
        (if (= quantity 1)
            thing
            (combine (together-copies-of combine
                        (- quantity 1)
                        thing)
                        thing)))))

(define stack-copies-of
    (lambda (quantity image)
        (together-copies-of stack quantity image)))
(define together-copies-of
  (lambda (combine quantity thing)
    (if (= quantity 1)
      thing
      (combine (together-copies-of combine
                (- quantity 1) thing)
               thing)))))

(define stack-copies-of
  (lambda (quantity image)
    (together-copies-of stack quantity image)))

Actual parameter
(define together-copies-of
  (lambda (combine quantity thing)
    (if (= quantity 1)
      thing
      (combine (together-copies-of combine
        (- quantity 1)
        thing)
        thing))))

(define stack-copies-of
  (lambda (quantity image)
    (together-copies-of stack quantity image)))
(define together-copies-of
  (lambda (combine quantity thing)
    (if (= quantity 1)
        thing
        (combine (together-copies-of combine
                      (- quantity 1)
                      thing)
                     thing)))))

(define stack-copies-of
  (lambda (quantity image)
    (together-copies-of stack quantity image)))

What is the definition of power?
(define together-copies-of
  (lambda (combine quantity thing)
    (if (= quantity 1)
      thing
      (combine (together-copies-of combine
        (- quantity 1)
        thing)
      thing)))))

(define stack-copies-of
  (lambda (quantity image)
    (together-copies-of stack quantity image)))

(define power
  (lambda (base exponent)
    (together-copies-of * exponent base)))
What is the definition of `num-odd-digits`?
((define num-digits-in-satisfying
  (lambda (n test?)
    (cond ((< n 0)
           (num-digits-in-satisfying (- n) test?))
           ((< n 10)
           (if (test? n) 1 0))
           ((test? (remainder n 10))
           (+ (num-digits-in-satisfying (quotient n 10) test?)
               1))
           (else
           (num-digits-in-satisfying (quotient n 10) test?))))))

(define num-odd-digits
  (lambda (n)
    (num-digits-in-satisfying n odd?)))
What is the definition of `num-6s`?
(define num-digits-in-satisfying
  (lambda (n test?)
    (cond ((< n 0)
            (num-digits-in-satisfying (- n) test?))
          ((< n 10)
           (if (test? n) 1 0))
          ((test? (remainder n 10))
           (+ (num-digits-in-satisfying (quotient n 10) test?)
               1))
          (else
           (num-digits-in-satisfying (quotient n 10) test?)))))

(define num-odd-digits
  (lambda (n)
    (num-digits-in-satisfying n odd?)))

(define num-6s
  (lambda (n)
    (num-digits-in-satisfying n (lambda (m) (= m 6)))))
The Halting Problem
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Is it possible to write a program that does halt, that can determine whether any other program would halt if it were executed?
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Is it possible to write a program that does halt, that can determine whether any other program would halt if it were executed?

NO!
The Halting Problem

(define return-seven
  (lambda ()
    7))
The Halting Problem

(define return-seven
  (lambda ()
    7))

(define loop-forever
  (lambda ()
    (loop-forever)))
The Halting Problem

(define return-seven
  (lambda ()
    7))

(define loop-forever
  (lambda ()
    (loop-forever)))

(define halts?
  (lambda (alpha)
    #t ; Bug. Should return #t if alpha halts, otherwise #f))
The Halting Problem

(define return-seven
  (lambda ()
    7))

(define loop-forever
  (lambda ()
    (loop-forever)))

(define halts?
  (lambda (alpha)
    #t ; Bug. Should return #t if alpha halts, otherwise #f))

> (halts? return-seven)
#t
> (halts? loop-forever)
#f
The Halting Problem

Proof by contradiction. Assume it is possible to write \texttt{halts?}, and show that assumption leads to a contradiction.
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Construct function \textit{debunk-halts?}
The Halting Problem

Proof by contradiction. Assume it is possible to write \texttt{halts?}, and show that assumption leads to a contradiction.

Construct function \texttt{debunk-halts}?

\[
\begin{align*}
&(\text{define debunk-halts?}) \\
&(\lambda () \\
&\quad (\text{if} (\text{halts? debunk-halts?}) \\
&\quad (\text{loop-forever}) \\
&\quad (\text{return-seven}))))
\end{align*}
\]
There are two possibilities:

```
(define debunk-halts?
    (lambda ()
        (if (halts? debunk-halts?)
            (loop-forever)
            (return-seven))))
```
There are two possibilities:
(a) debunk-halts? halts
There are two possibilities:

(a) `debunk-halts? halts`

\[ \Rightarrow (\text{halts? debunk-halts?}) \text{ returns } \#t \]
There are two possibilities:
(a) `debunk-halts? halts`  
   ⇒ `(halts? debunk-halts?) returns #t`  
   ⇒ `loop-forever executes`
There are two possibilities:

(a) `debunk-halts? halts`

⇒ `(halts? debunk-halts?)` *returns* `#t`

⇒ `loop-forever` *executes*

⇒ `debunk-halts?` *does not halt*
(define debunk-halts?
  (lambda ()
    (if (halts? debunk-halts?)
      (loop-forever)
      (return-seven))))

There are two possibilities:
(a) debunk-halts? halts
    ⇒ (halts? debunk-halts?) returns #t
    ⇒ loop-forever executes
    ⇒ debunk-halts? does not halt
Contradiction
(define debunk-halts?
  (lambda ()
    (if (halts? debunk-halts?)
      (loop-forever)
      (return-seven))))

There are two possibilities:
(b) debunk-halts? does not halt
(define debunk-halts? 
   (lambda () 
      (if (halts? debunk-halts?) 
          (loop-forever) 
          (return-seven))))

There are two possibilities:
(b) debunk-halts? does not halt
   ⇒ (halts? debunk-halts?) returns #f
(define debunk-halts?
  (lambda ()
    (if (halts? debunk-halts?)
      (loop-forever)
      (return-seven))))

There are two possibilities:
(b) debunk-halts? does not halt
   ⇒ (halts? debunk-halts?) returns #f
   ⇒ return-seven executes
There are two possibilities:

(b) `debunk-halts?` does not halt

⇒ `(halts? debunk-halts?)` returns `#f`

⇒ `return-seven` executes

⇒ `debunk-halts?` does halt
(define debunk-halts?
    (lambda ()
        (if (halts? debunk-halts?)
            (loop-forever)
            (return-seven)))))

There are two possibilities:

(b) debunk-halts? does not halt

⇒ (halts? debunk-halts?) returns #f

⇒ return-seven executes

⇒ debunk-halts? does halt

Contradiction
(define debunk-halts?
  (lambda ()
    (if (halts? debunk-halts?)
      (loop-forever)
      (return-seven))))

Conclusion:
The assumption implies a contradiction in all possible scenarios.
Therefore, the assumption is false, and it is impossible to write halts?
Procedure factories

> (double 4)
  8
> (double 5)
  10
> (triple 4)
  12
> (triple 5)
  15
Procedure factories

> (double 4)  
8  
> (double 5)  
10  
> (triple 4)  
12  
> (triple 5)  
15

(define double  
  (make-multiplier 2))

(define triple  
  (make-multiplier 3))
Procedure factories

(define double
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(define double
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(define triple
  (make-multiplier 3))

Define make-multiplier

(define make-multiplier
  (lambda (scaling-factor)
Procedure factories

(define double
  (make-multiplier 2))

(define triple
  (make-multiplier 3))

Define make-multiplier

(define make-multiplier
  (lambda (scaling-factor)
    (lambda (x)
      (* x scaling-factor))
Procedure factories

(define double
  (make-multiplier 2))

(define triple
  (make-multiplier 3))

Define make-multiplier

(define make-multiplier
  (lambda (scaling-factor)
    (lambda (x)
      (* x scaling-factor)))))
If the factory manufactures a function that calls itself the function must be named.

```
(define function-factory
  (lambda (parameter-for-factory)
    (define function-returned
      (lambda (parameter-for-function-returned)
        ...
        recursive call to function-returned
        ...
        )
    function-returned))
```
> (repeatedly-square 2 0)
  2
> (repeatedly-square 2 1)
  4
> (repeatedly-square 2 2)
  16
> (repeatedly-square 2 3)
  256
> (repeatedly-square 2 0)
2
> (repeatedly-square 2 1)
4
> (repeatedly-square 2 2)
16
> (repeatedly-square 2 3)
256

Function returned by the factory

(repeatedly-square 2 3)
> (repeatedly-square 2 0)
2
> (repeatedly-square 2 1)
4
> (repeatedly-square 2 2)
16
> (repeatedly-square 2 3)
256

Function returned by the factory

(repeatedly-square 2 3)

Two parameters
> (repeatedly-square 2 0)
2
> (repeatedly-square 2 1)
4
> (repeatedly-square 2 2)
16
> (repeatedly-square 2 3)
256

Function returned by the factory

(repeatedly-square 2 3)

Two parameters

Thing on which to operate
> (repeatedly-square 2 0)
  2
> (repeatedly-square 2 1)
  4
> (repeatedly-square 2 2)
  16
> (repeatedly-square 2 3)
  256

Function returned by the factory

(repeatedly-square 2 3)

Two parameters

Thing on which to operate

How many times to operate
> (repeatedly-square 2 3)
256

Function returned by the factory

(repeatedly-square 2 3)

Calling the factory to make the function
> (repeatedly-square 2 3)
256

Function returned by the factory

(repeatedly-square 2 3)

Calling the factory to make the function

(define repeatedly-square
  (make-repeated-version-of sqr))
> (repeatedly-square 2 3)
256

Function returned by the factory

(repeatedly-square 2 3)

Calling the factory to make the function

(define repeatedly-square
  (make-repeated-version-of sqr))

Returns a function having two parameters
> (repeatedly-square 2 3)
256

Function returned by the factory

(repeatedly-square 2 3)

Calling the factory to make the function

(define repeatedly-square
  (make-repeated-version-of sqr))

Returns a function having two parameters

Has only one parameter itself, the operation
(repeatedly-square 2 3)

(define repeatedly-square
 (make-repeated-version-of sqr))
(repeatedly-square 2 3)

(define repeatedly-square
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Define the factory
(repeatedly-square 2 3)

(define repeatedly-square
  (make-repeated-version-of sqr))

Define the factory

(define make-repeated-version-of
(repeatedly-square 2 3)

(define repeatedly-square
  (make-repeated-version-of sqr))

Define the factory

(define make-repeated-version-of
  (lambda (f) ; make a repeated version of f
(repeatedly-square 2 3)

(define repeatedly-square
  (make-repeated-version-of sqr))

Define the factory

(define make-repeated-version-of
  (lambda (f)  ; make a repeated version of f
    (define the-repeated-version
      (lambda (b n)
        (if (= n 0)
            b
            (the-repeated-version (f b) (- n 1))))))))
(repeatedly-square 2 3)

(define repeatedly-square
  (make-repeated-version-of sqr))

Define the factory

(define make-repeated-version-of
  (lambda (f) ; make a repeated version of f
    (define the-repeated-version
      (lambda (b n) ; which does f n times to b
        ...)))
(repeatedly-square 2 3)

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Define the factory

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    (define the-repeated-version
      (lambda (b n) ; which does f n times to b
        (if (= n 0)
(repeatedly-square 2 3)

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