Higher-Order Procedures

In the functional paradigm, functions themselves can be processed as data.

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In the same way you can pass data values as parameters in a function, you can pass function as a parameter in another function.

```
(define power
  (lambda (b e)
    (if (= e 1)
        b
        (* (power b (- e 1)) b))))

(define stack-copies-of
  (lambda (quantity image)
    (if (= quantity 1)
        image
        (stack (stack-copies-of (- quantity 1) image) image))))
```

How are the functions similar?

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(define power
  (lambda (b e)
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How are the functions different?

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(define power
  (lambda (b e)
    (if (= e 1)
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(define stack-copies-of
  (lambda (quantity image)
    (if (= quantity 1)
        image
        (stack (stack-copies-of (- quantity 1) image) image))))
```

The form is the same. The functions are different.

```
(define together-copies-of
  (lambda (combine quantity thing)
    (if (= quantity 1)
        thing
        (combine (together-copies-of combine
                                     (- quantity 1)
                                     thing)
                 thing))))
```

```
(define together-copies-of
  (lambda (combine quantity thing)
    (if (= quantity 1)
        thing
        (combine (together-copies-of combine
                                     (- quantity 1)
                                     thing)
                 thing))))
```

The first parameter in the function `together-copies-of` is a function.

```
(define together-copies-of
  (lambda (combine quantity thing)
    (if (= quantity 1)
        thing
        (combine (together-copies-of combine
                                     (- quantity 1)
                                     thing)
                 thing))))


(define stack-copies-of
  (lambda (quantity image)
    (together-copies-of stack quantity image)))
```

```
(define together-copies-of
  (lambda (combine quantity thing)
    (if (= quantity 1)
        thing
        (combine (together-copies-of combine
                                     (- quantity 1)
                                     thing)
                 thing))))
```

```
(define stack-copies-of
  (lambda (quantity image)
    (together-copies-of stack quantity image)))
```

Actual parameter

```
(define together-copies-of
  (lambda (combine quantity thing)
    (if (= quantity 1)
        thing
        (combine (together-copies-of combine
                                     (- quantity 1)
                                     thing)
                 thing))))


(define stack-copies-of
  (lambda (quantity image)
    (together-copies-of stack quantity image)))
```

Formal parameter

```
(define together-copies-of
  (lambda (combine quantity thing)
    (if (= quantity 1)
        thing
        (combine (together-copies-of combine
                                     (- quantity 1)
                                     thing)
                 thing))))
```

```
(define stack-copies-of
  (lambda (quantity image)
    (together-copies-of stack quantity image)))
```

What is the definition of `power`?

```
(define together-copies-of
  (lambda (combine quantity thing)
    (if (= quantity 1)
        thing
        (combine (together-copies-of combine
                                     (- quantity 1)
                                     thing)
                 thing))))


(define stack-copies-of
  (lambda (quantity image)
    (together-copies-of stack quantity image)))


(define power
  (lambda (base exponent)
    (together-copies-of * exponent base)))
```

```
(define num-digits-in-satisfying
  (lambda (n test?)
    (cond ((< n 0)
           (num-digits-in-satisfying (- n) test?))
          ((< n 10)
           (if (test? n) 1 0))
          ((test? (remainder n 10))
           (+ (num-digits-in-satisfying (quotient n 10) test?)
              1))
          (else
           (num-digits-in-satisfying (quotient n 10) test?)))))
```

What is the definition of `num-odd-digits`?

```
(define num-digits-in-satisfying
  (lambda (n test?)
    (cond ((< n 0)
           (num-digits-in-satisfying (- n) test?))
          ((< n 10)
           (if (test? n) 1 0))
          ((test? (remainder n 10))
           (+ (num-digits-in-satisfying (quotient n 10) test?)
              1))
          (else
           (num-digits-in-satisfying (quotient n 10) test?)))))

(define num-odd-digits
  (lambda (n)
    (num-digits-in-satisfying n odd?)))
```

```
(define num-digits-in-satisfying
  (lambda (n test?)
    (cond ((< n 0)
           (num-digits-in-satisfying (- n) test?))
          ((< n 10)
           (if (test? n) 1 0))
          ((test? (remainder n 10))
           (+ (num-digits-in-satisfying (quotient n 10) test?)
              1))
          (else
           (num-digits-in-satisfying (quotient n 10) test?)))))

(define num-odd-digits
  (lambda (n)
    (num-digits-in-satisfying n odd?)))
```

What is the definition of `num-6s`?

```
(define num-digits-in-satisfying
  (lambda (n test?)
    (cond ((< n 0)
           (num-digits-in-satisfying (- n) test?))
          ((< n 10)
           (if (test? n) 1 0))
          ((test? (remainder n 10))
           (+ (num-digits-in-satisfying (quotient n 10) test?)
              1))
          (else
           (num-digits-in-satisfying (quotient n 10) test?)))))

(define num-odd-digits
  (lambda (n)
    (num-digits-in-satisfying n odd?)))

(define num-6s
  (lambda (n)
    (num-digits-in-satisfying n (lambda (m) (= m 6)))))
```

The Halting Problem

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Is it possible to write a program that does halt, that can determine whether any other program would halt if it were executed?

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NO!

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```
(define return-seven
  (lambda ()
    7))
```

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```
(define return-seven
  (lambda ()
    7))

(define loop-forever
  (lambda ()
    (loop-forever)))
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(define return-seven
  (lambda ()
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(define loop-forever
  (lambda ()
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(define halts?
  (lambda (alpha)
    #t ; Bug. Should return #t if alpha halts, otherwise #f
    ))
```

The Halting Problem

```
(define return-seven
  (lambda ()
    7))

(define loop-forever
  (lambda ()
    (loop-forever)))

(define halts?
  (lambda (alpha)
    #t ; Bug. Should return #t if alpha halts, otherwise #f
    ))


> (halts? return-seven)
#t
> (halts? loop-forever)
#f
```

The Halting Problem

Proof by contradiction. Assume it is possible to write `halts?`, and show that assumption leads to a contradiction.

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Construct function `debunk-halts?`

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Construct function `debunk-halts?`

```
(define debunk-halts?
  (lambda ()
    (if (halts? debunk-halts?)
        (loop-forever)
        (return-seven))))
```

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(define debunk-halts?
  (lambda ()
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```

There are two possibilities:

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(define debunk-halts?
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```

There are two possibilities:

(a) `debunk-halts?` halts

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(define debunk-halts?
  (lambda ()
    (if (halts? debunk-halts?)
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```

There are two possibilities:

(a) `debunk-halts?` halts

⇒ `(halts? debunk-halts?)` returns `#t`

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⇒ `(halts? debunk-halts?)` returns `#t`

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⇒ `debunk-halts?` does not halt

```
(define debunk-halts?
  (lambda ()
    (if (halts? debunk-halts?)
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```

There are two possibilities:

(a) `debunk-halts?` halts

$\Rightarrow$ `(halts? debunk-halts?)` returns `#t`

$\Rightarrow$ `loop-forever` executes

$\Rightarrow$ `debunk-halts?` does not halt

Contradiction

```
(define debunk-halts?
  (lambda ()
    (if (halts? debunk-halts?)
        (loop-forever)
        (return-seven))))
```

There are two possibilities:

(b) `debunk-halts?` does not halt

```
(define debunk-halts?
  (lambda ()
    (if (halts? debunk-halts?)
        (loop-forever)
        (return-seven))))
```

There are two possibilities:

(b) `debunk-halts?` does not halt

⇒ `(halts? debunk-halts?)` returns `#f`

```
(define debunk-halts?
  (lambda ()
    (if (halts? debunk-halts?)
        (loop-forever)
        (return-seven))))
```

There are two possibilities:

(b) `debunk-halts?` does not halt

⇒ `(halts? debunk-halts?)` returns `#f`

⇒ `return-seven` executes

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(define debunk-halts?
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        (return-seven))))
```

There are two possibilities:

(b) `debunk-halts?` does not halt

⇒ `(halts? debunk-halts?)` returns `#f`

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⇒ `debunk-halts?` does halt

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    (if (halts? debunk-halts?)
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```

Conclusion:

The assumption implies a contradiction in all possible scenarios.
Therefore, the assumption is false, and it is impossible to write `halts?`

Procedure factories

```
> (double 4)
8
> (double 5)
10
> (triple 4)
12
> (triple 5)
15
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> (double 5)
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```
(define double
  (make-multiplier 2))

(define triple
  (make-multiplier 3))
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Procedure factories

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(define double
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Define `make-multiplier`

```
(define make-multiplier
  (lambda (scaling-factor)
```

Procedure factories

```
(define double
  (make-multiplier 2))

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  (make-multiplier 3))
```

Define `make-multiplier`

```
(define make-multiplier
  (lambda (scaling-factor)
    (lambda (x)
```

Procedure factories

```
(define double
  (make-multiplier 2))

(define triple
  (make-multiplier 3))
```

Define `make-multiplier`

```
(define make-multiplier
  (lambda (scaling-factor)
    (lambda (x)
      (* x scaling-factor))))
```

If the factory manufactures a function that calls itself the function must be named.

```
(define function-factory
  (lambda (parameter-for-factory)
    (define function-returned
      (lambda (parameter-for-function-returned)
           ...
             recursive call to function-returned
           ...                                      ))
    function-returned))
```

```
> (repeatedly-square 2 0)
2
> (repeatedly-square 2 1)
4
> (repeatedly-square 2 2)
16
> (repeatedly-square 2 3)
256
```

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> (repeatedly-square 2 0)
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256
```

Function returned by the factory

```
(repeatedly-square 2 3)
```

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Two parameters

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(repeatedly-square 2 3)
```

Two parameters

Thing on which to operate

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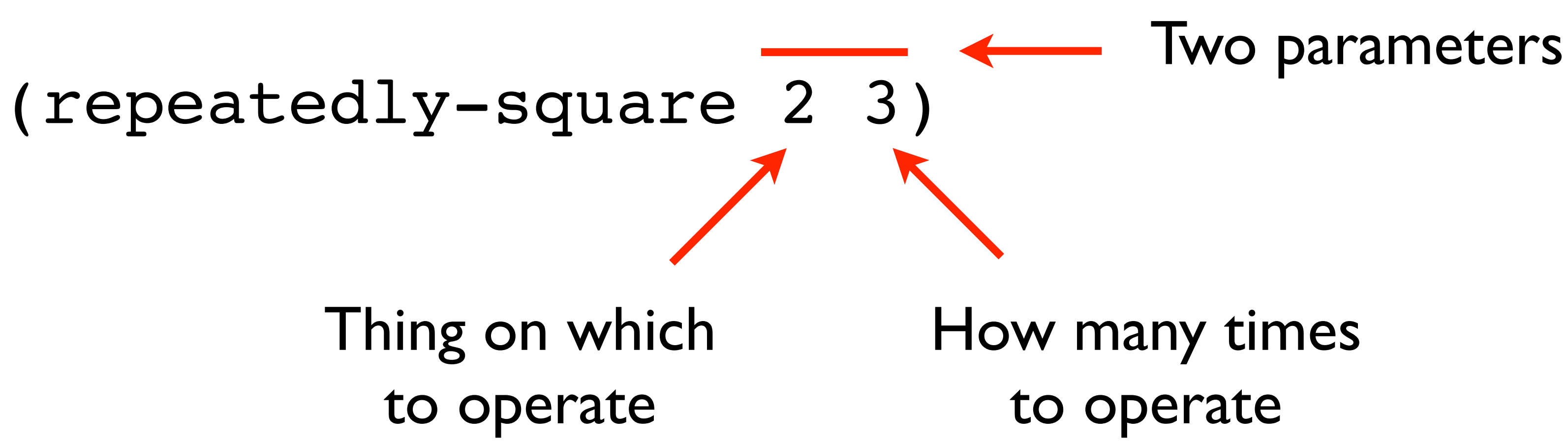
Function returned by the factory



```
> (repeatedly-square 2 3)
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Function returned by the factory

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(repeatedly-square 2 3)
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Calling the factory to make the function

```
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Function returned by the factory

```
(repeatedly-square 2 3)
```

Calling the factory to make the function

```
(define repeatedly-square
  (make-repeated-version-of sqr))
```

```
> (repeatedly-square 2 3)
256
```

Function returned by the factory

```
(repeatedly-square 2 3)
```

Calling the factory to make the function

```
(define repeatedly-square
  (make-repeated-version-of sqr))
```

Returns a function
having two parameters

```
> (repeatedly-square 2 3)
256
```

Function returned by the factory

```
(repeatedly-square 2 3)
```

Calling the factory to make the function

```
(define repeatedly-square
  (make-repeated-version-of sqr))
```

Returns a function having two parameters (make-repeated-version-of)

Has only one parameter itself, the operation (sqr)

```
(repeatedly-square 2 3)

(define repeatedly-square
  (make-repeated-version-of sqr))
```

```
(repeatedly-square 2 3)

(define repeatedly-square
  (make-repeated-version-of sqr))
```

Define the factory

```
(repeatedly-square 2 3)

(define repeatedly-square
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```

Define the factory

```
(define make-repeated-version-of
```

```
(repeatedly-square 2 3)

(define repeatedly-square
  (make-repeated-version-of sqr))
```

Define the factory

```
(define make-repeated-version-of
  (lambda (f)  ; make a repeated version of f
```

```
(repeatedly-square 2 3)

(define repeatedly-square
  (make-repeated-version-of sqr))
```

Define the factory

```
(define make-repeated-version-of
  (lambda (f)  ; make a repeated version of f
    (define the-repeated-version
```

```
(repeatedly-square 2 3)

(define repeatedly-square
  (make-repeated-version-of sqr))
```

Define the factory

```
(define make-repeated-version-of
  (lambda (f)  ; make a repeated version of f
    (define the-repeated-version
      (lambda (b n)  ; which does f n times to b
```

```
(repeatedly-square 2 3)

(define repeatedly-square
  (make-repeated-version-of sqr))
```

Define the factory

```
(define make-repeated-version-of
  (lambda (f)  ; make a repeated version of f
    (define the-repeated-version
      (lambda (b n)  ; which does f n times to b
        (if (= n 0)
```

```
(repeatedly-square 2 3)

(define repeatedly-square
  (make-repeated-version-of sqr))
```

Define the factory

```
(define make-repeated-version-of
  (lambda (f)  ; make a repeated version of f
    (define the-repeated-version
      (lambda (b n)  ; which does f n times to b
        (if (= n 0)
            b
```

```
(repeatedly-square 2 3)

(define repeatedly-square
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Define the factory

```
(define make-repeated-version-of
  (lambda (f)  ; make a repeated version of f
    (define the-repeated-version
      (lambda (b n)  ; which does f n times to b
        (if (= n 0)
            b
            (the-repeated-version (f b) (- n 1)))))
```

```
(repeatedly-square 2 3)

(define repeatedly-square
  (make-repeated-version-of sqr))
```

Define the factory

```
(define make-repeated-version-of
  (lambda (f)  ; make a repeated version of f
    (define the-repeated-version
      (lambda (b n)  ; which does f n times to b
        (if (= n 0)
            b
            (the-repeated-version (f b) (- n 1)))))
    the-repeated-version))
```