Theorems from CDS4LTL

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Abstract

The first section of this document is a collection of the axioms and theorems of the propositional calculus in Gries and Schneider's book *A Logical Approach to Discrete Math*, Springer-Verlag, 1993 (LADM). The numbering is consistent with that text with the chapter number followed by the equation number separated by a period. Additional theorems, either not included in LADM or included but not numbered, are indicated by a three-part number with two period separators. The second section is a collection of the axioms and theorems of linear temporal logic in Warford, Vega, and Staley's paper *A Calculational Deductive System for Linear Temporal Logic* (CDS4LTL), ACM Computing Surveys, Vol. 53, No. 3, June 2020.

Table of Precedences

$[x := e]$ (textual substitution)	Highest precedence
$\neg \quad \circ \quad \Diamond \quad \Box$	
$\mathcal{U} \quad \mathcal{W}$	
$=$ (conjunctional)	
$\vee \quad \wedge$	
$\Rightarrow \quad \Leftarrow$	
$\equiv$ (associative)	Lowest precedence

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Theorems of the Propositional Calculus

Equivalence and $true$

(3.1) **Axiom, Associativity of $\equiv$:** $((p \equiv q) \equiv r) \;\equiv\; (p \equiv (q \equiv r))$
(3.2) **Axiom, Symmetry of $\equiv$:** $p \equiv q \equiv q \equiv p$
(3.3) **Axiom, Identity of $\equiv$:** $true \equiv q \equiv q$
(3.4) $true$
(3.5) **Reflexivity of $\equiv$:** $p \equiv p$

Negation, inequivalence, and $false$

(3.8) **Definition of $false$:** $false \equiv \neg true$
(3.9) **Axiom, Distributivity of $\neg$ over $\equiv$:** $\neg(p \equiv q) \;\equiv\; \neg p \equiv q$
(3.10) **Definition of $\not\equiv$:** $(p \not\equiv q) \;\equiv\; \neg(p \equiv q)$
(3.11) $\neg p \equiv q \equiv p \equiv \neg q$
(3.12) **Double negation:** $\neg\neg p \equiv p$
(3.13) **Negation of $false$:** $\neg false \equiv true$
(3.14) $(p \not\equiv q) \;\equiv\; \neg p \equiv q$
(3.15) $\neg p \equiv p \equiv false$
(3.16) **Symmetry of $\not\equiv$:** $(p \not\equiv q) \;\equiv\; (q \not\equiv p)$
(3.17) **Associativity of $\not\equiv$:** $((p \not\equiv q) \not\equiv r) \;\equiv\; (p \not\equiv (q \not\equiv r))$
(3.18) **Mutual associativity:** $((p \not\equiv q) \equiv r) \;\equiv\; (p \not\equiv (q \equiv r))$
(3.19) **Mutual interchangeability:** $p \not\equiv q \equiv r \quad\equiv\quad p \equiv q \not\equiv r$
(3.19.1) $p \not\equiv p \not\equiv q \;\equiv\; q$

Disjunction

(3.24) **Axiom, Symmetry of $\vee$:** $p \vee q \equiv q \vee p$
(3.25) **Axiom, Associativity of $\vee$:** $(p \vee q) \vee r \equiv p \vee (q \vee r)$
(3.26) **Axiom, Idempotency of $\vee$:** $p \vee p \equiv p$
(3.27) **Axiom, Distributivity of $\vee$ over $\equiv$:** $p \vee (q \equiv r) \equiv p \vee q \equiv p \vee r$
(3.28) **Axiom, Excluded middle:** $p \vee \neg p$
(3.29) **Zero of $\vee$:** $p \vee true \equiv true$
(3.30) **Identity of $\vee$:** $p \vee false \equiv p$
(3.31) **Distributivity of $\vee$ over $\vee$:** $p \vee (q \vee r) \equiv (p \vee q) \vee (p \vee r)$
(3.32) $p \vee q \equiv p \vee \neg q \equiv p$

Conjunction

(3.35) **Axiom, Golden rule:** $p \wedge q \;\equiv\; p \;\equiv\; q \;\equiv\; p \vee q$

(3.36) **Symmetry of $\wedge$:** $p \wedge q \equiv q \wedge p$

(3.37) **Associativity of $\wedge$:** $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

(3.38) **Idempotency of $\wedge$:** $p \wedge p \equiv p$

(3.39) **Identity of $\wedge$:** $p \wedge true \equiv p$

(3.40) **Zero of $\wedge$:** $p \wedge false \equiv false$

(3.41) **Distributivity of $\wedge$ over $\wedge$:** $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge (p \wedge r)$

(3.42) **Contradiction:** $p \wedge \neg p \equiv false$

(3.43) **Absorption:**
(a) $p \wedge (p \vee q) \equiv p$
(b) $p \vee (p \wedge q) \equiv p$

(3.44) **Absorption:**
(a) $p \wedge (\neg p \vee q) \equiv p \wedge q$
(b) $p \vee (\neg p \wedge q) \equiv p \vee q$

(3.45) **Distributivity of $\vee$ over $\wedge$:** $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

(3.46) **Distributivity of $\wedge$ over $\vee$:** $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

(3.47) **De Morgan:**
(a) $\neg(p \wedge q) \equiv \neg p \vee \neg q$
(b) $\neg(p \vee q) \equiv \neg p \wedge \neg q$

(3.48) $p \wedge q \;\equiv\; p \wedge \neg q \;\equiv\; \neg p$

(3.49) $p \wedge (q \equiv r) \;\equiv\; p \wedge q \;\equiv\; p \wedge r \;\equiv\; p$

(3.50) $p \wedge (q \equiv p) \;\equiv\; p \wedge q$

(3.51) **Replacement:** $(p \equiv q) \wedge (r \equiv p) \quad\equiv\quad (p \equiv q) \wedge (r \equiv q)$

(3.52) **Equivalence:** $p \equiv q \;\equiv\; (p \wedge q) \vee (\neg p \wedge \neg q)$

(3.53) **Exclusive or:** $p \not\equiv q \;\equiv\; (\neg p \wedge q) \vee (p \wedge \neg q)$

(3.55) $(p \wedge q) \wedge r \;\equiv\; p \;\equiv\; q \;\equiv\; r \;\equiv\; p \vee q \;\equiv\; q \vee r \;\equiv\; r \vee p \;\equiv\; p \vee q \vee r$

Implication

(3.57) **Definition of Implication:** $p \Rightarrow q \;\equiv\; p \vee q \;\equiv\; q$

(3.58) **Axiom, Consequence:** $p \Leftarrow q \;\equiv\; q \Rightarrow p$

(3.59) **Implication:** $p \Rightarrow q \;\equiv\; \neg p \vee q$

(3.60) **Implication:** $p \Rightarrow q \;\equiv\; p \wedge q \;\equiv\; p$

(3.61) **Contrapositive:** $p \Rightarrow q \;\equiv\; \neg q \Rightarrow \neg p$

(3.62) $p \Rightarrow (q \equiv r) \;\equiv\; p \land q \;\equiv\; p \land r$

(3.63) **Distributivity of $\Rightarrow$ over $\equiv$:** $p \Rightarrow (q \equiv r) \;\equiv\; (p \Rightarrow q) \;\equiv\; (p \Rightarrow r)$

(3.63.1) **Distributivity of $\Rightarrow$ over $\land$:** $p \Rightarrow q \land r \;\equiv\; (p \Rightarrow q) \land (p \Rightarrow r)$

(3.63.2) **Distributivity of $\Rightarrow$ over $\lor$:** $p \Rightarrow q \lor r \;\equiv\; (p \Rightarrow q) \lor (p \Rightarrow r)$

(3.64) $p \Rightarrow (q \Rightarrow r) \;\equiv\; (p \Rightarrow q) \Rightarrow (p \Rightarrow r)$

(3.65) **Shunting:** $p \land q \Rightarrow r \;\equiv\; p \Rightarrow (q \Rightarrow r)$

(3.66) $p \land (p \Rightarrow q) \;\equiv\; p \land q$

(3.67) $p \land (q \Rightarrow p) \;\equiv\; p$

(3.68) $p \lor (p \Rightarrow q) \;\equiv\; true$

(3.69) $p \lor (q \Rightarrow p) \;\equiv\; q \Rightarrow p$

(3.70) $p \lor q \Rightarrow p \land q \;\equiv\; p \;\equiv\; q$

(3.71) **Reflexivity of $\Rightarrow$:** $p \Rightarrow p$

(3.72) **Right zero of $\Rightarrow$:** $p \Rightarrow true \;\equiv\; true$

(3.73) **Left identity of $\Rightarrow$:** $true \Rightarrow p \;\equiv\; p$

(3.74) $p \Rightarrow false \;\equiv\; \neg p$

(3.74.1) $\neg p \Rightarrow false \;\equiv\; p$

(3.75) $false \Rightarrow p \;\equiv\; true$

(3.76) **Weakening/strengthening:**

(a) $p \Rightarrow p \lor q$ (Weakening the consequent)

(b) $p \land q \Rightarrow p$ (Strengthening the antecedent)

(c) $p \land q \Rightarrow p \lor q$ (Weakening/strengthening)

(d) $p \lor (q \land r) \Rightarrow p \lor q$

(e) $p \land q \Rightarrow p \land (q \lor r)$

(3.76.1) $p \land q \Rightarrow p \lor r$ (Weakening/strengthening)

(3.76.2) $(p \Rightarrow q) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \Rightarrow r))$

(3.76.3) $(p \lor q) \land (q \Rightarrow r) \Rightarrow p \lor r$

(3.77) **Modus ponens:** $p \land (p \Rightarrow q) \Rightarrow q$

(3.77.1) **Modus tollens:** $(p \Rightarrow q) \land \neg q \Rightarrow \neg p$

(3.77.2) $((p \Rightarrow q) \Rightarrow (r \Rightarrow s)) \land (s \Rightarrow t) \Rightarrow ((p \Rightarrow q) \Rightarrow (r \Rightarrow t))$

(3.77.3) $((p \Rightarrow (q \Rightarrow r)) \land (r \Rightarrow s)) \Rightarrow (p \Rightarrow (q \Rightarrow s))$

(3.78) $(p \Rightarrow r) \land (q \Rightarrow r) \;\equiv\; p \lor q \Rightarrow r$

(3.78.1) $(p \Rightarrow r) \lor (q \Rightarrow r) \;\equiv\; p \land q \Rightarrow r$

(3.79) $(p \Rightarrow r) \land (\neg p \Rightarrow r) \;\equiv\; r$

(3.80) **Mutual implication:** $(p \Rightarrow q) \land (q \Rightarrow p) \;\equiv\; (p \equiv q)$

(3.81) **Antisymmetry:** $(p \Rightarrow q) \land (q \Rightarrow p) \Rightarrow (p \equiv q)$

(3.82) **Transitivity:**

(a) $(p \Rightarrow q) \land (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$

(b) $(p \equiv q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$
(c) $(p \Rightarrow q) \wedge (q \equiv r) \Rightarrow (p \Rightarrow r)$

(3.82.1) **Transitivity of $\equiv$:** $(p \equiv q) \wedge (q \equiv r) \Rightarrow (p \equiv r)$

(3.82.2) $(p \equiv q) \Rightarrow (p \Rightarrow q)$

Leibniz as an axiom

This section uses the following notation: E^z_X means $E[z := X]$.

(3.83) **Axiom, Leibniz:** $e = f \Rightarrow E^z_e = E^z_f$

(3.84) **Substitution:**
(a) $(e = f) \wedge E^z_e \;\equiv\; (e = f) \wedge E^z_f$
(b) $(e = f) \Rightarrow E^z_e \;\equiv\; (e = f) \Rightarrow E^z_f$
(c) $q \wedge (e = f) \Rightarrow E^z_e \;\equiv\; q \wedge (e = f) \Rightarrow E^z_f$

(3.85) **Replace by $true$:**
(a) $p \Rightarrow E^z_p \;\equiv\; p \Rightarrow E^z_{true}$
(b) $q \wedge p \Rightarrow E^z_p \;\equiv\; q \wedge p \Rightarrow E^z_{true}$

(3.86) **Replace by $false$:**
(a) $E^z_p \Rightarrow p \;\equiv\; E^z_{false} \Rightarrow p$
(b) $E^z_p \Rightarrow p \vee q \;\equiv\; E^z_{false} \Rightarrow p \vee q$

(3.87) **Replace by $true$:** $p \wedge E^z_p \;\equiv\; p \wedge E^z_{true}$

(3.88) **Replace by $false$:** $p \vee E^z_p \;\equiv\; p \vee E^z_{false}$

(3.89) **Shannon:** $E^z_p \;\equiv\; (p \wedge E^z_{true}) \vee (\neg p \wedge E^z_{false})$

(3.89.1) $E^z_{true} \wedge E^z_{false} \Rightarrow E^z_p$

Additional theorems concerning implication

(4.1) $p \Rightarrow (q \Rightarrow p)$

(4.2) **Monotonicity of $\vee$:** $(p \Rightarrow q) \Rightarrow (p \vee r \Rightarrow q \vee r)$

(4.3) **Monotonicity of $\wedge$:** $(p \Rightarrow q) \Rightarrow (p \wedge r \Rightarrow q \wedge r)$

Proof technique metatheorems.

(4.4) **Deduction (assume conjuncts of antecedent):**
To prove $P_1 \wedge P_2 \Rightarrow Q$, assume P_1 and P_2, and prove Q.
You cannot use textual substitution in P_1 or P_2.

(4.7) **Mutual implication:** To prove $P \equiv Q$, prove $P \Rightarrow Q$ and $Q \Rightarrow P$.

(4.7.1) **Truth implication:** To prove P, prove $true \Rightarrow P$.

(4.9) **Proof by contradiction:** To prove P, prove $\neg P \Rightarrow false$.

(4.12) **Proof by contrapositive:** To prove $P \Rightarrow Q$, prove $\neg Q \Rightarrow \neg P$.

Theorems of Linear Temporal Logic

Next $\circ$

(1) **Axiom, Self-dual:** $\circ \neg p \equiv \neg \circ p$

(2) **Axiom, Distributivity of $\circ$ over $\Rightarrow$:** $\circ (p \Rightarrow q) \equiv \circ p \Rightarrow \circ q$

(3) **Linearity:** $\circ p \equiv \neg \circ \neg p$

(4) **Distributivity of $\circ$ over $\vee$:** $\circ (p \vee q) \equiv \circ p \vee \circ q$

(5) **Distributivity of $\circ$ over $\wedge$:** $\circ (p \wedge q) \equiv \circ p \wedge \circ q$

(6) **Distributivity of $\circ$ over $\equiv$:** $\circ (p \equiv q) \equiv \circ p \equiv \circ q$

(7) **Truth of $\circ$:** $\circ true \equiv true$

(8) **Falsehood of $\circ$:** $\circ false \equiv false$

Until $\mathcal{U}$

(9) **Axiom, Distributivity of $\circ$ over $\mathcal{U}$:** $\circ (p \,\mathcal{U}\, q) \equiv \circ p \,\mathcal{U} \circ q$

(10) **Axiom, Expansion of $\mathcal{U}$:** $p \,\mathcal{U}\, q \equiv q \vee (p \wedge \circ (p \,\mathcal{U}\, q))$

(11) **Axiom, Right zero of $\mathcal{U}$:** $p \,\mathcal{U}\, false \equiv false$

(12) **Axiom, Left distributivity of $\mathcal{U}$ over $\vee$:** $p \,\mathcal{U}\, (q \vee r) \equiv p \,\mathcal{U}\, q \vee p \,\mathcal{U}\, r$

(13) **Axiom, Right distributivity of $\mathcal{U}$ over $\vee$:** $p \,\mathcal{U}\, r \vee q \,\mathcal{U}\, r \Rightarrow (p \vee q) \,\mathcal{U}\, r$

(14) **Axiom, Left distributivity of $\mathcal{U}$ over $\wedge$:** $p \,\mathcal{U}\, (q \wedge r) \Rightarrow p \,\mathcal{U}\, q \wedge p \,\mathcal{U}\, r$

(15) **Axiom, Right distributivity of $\mathcal{U}$ over $\wedge$:** $(p \wedge q) \,\mathcal{U}\, r \equiv p \,\mathcal{U}\, r \wedge q \,\mathcal{U}\, r$

(16) **Axiom, $\mathcal{U}$ implication ordering:** $p \,\mathcal{U}\, q \wedge \neg q \,\mathcal{U}\, r \Rightarrow p \,\mathcal{U}\, r$

(17) **Axiom, Right $\mathcal{U} \vee$ ordering:** $p \,\mathcal{U}\, (q \,\mathcal{U}\, r) \Rightarrow (p \vee q) \,\mathcal{U}\, r$

(18) **Axiom, Right $\wedge \mathcal{U}$ ordering:** $p \,\mathcal{U}\, (q \wedge r) \Rightarrow (p \,\mathcal{U}\, q) \,\mathcal{U}\, r$

(19) **Right distributivity of $\mathcal{U}$ over $\Rightarrow$:** $(p \Rightarrow q) \,\mathcal{U}\, r \Rightarrow (p \,\mathcal{U}\, r \Rightarrow q \,\mathcal{U}\, r)$

(20) **Right zero of $\mathcal{U}$:** $p \,\mathcal{U}\, true \equiv true$

(21) **Left identity of $\mathcal{U}$:** $false \,\mathcal{U}\, q \equiv q$

(22) **Idempotency of $\mathcal{U}$:** $p \,\mathcal{U}\, p \equiv p$

(23) **$\mathcal{U}$ excluded middle:** $p \,\mathcal{U}\, q \vee p \,\mathcal{U}\, \neg q$

(24) $\neg p \,\mathcal{U}\, (q \,\mathcal{U}\, r) \wedge p \,\mathcal{U}\, r \Rightarrow q \,\mathcal{U}\, r$

(25) $p \, \mathcal{U} \, (\neg q \, \mathcal{U} \, r) \wedge q \, \mathcal{U} \, r \Rightarrow p \, \mathcal{U} \, r$

(26) $p \, \mathcal{U} \, q \wedge \neg q \, \mathcal{U} \, p \Rightarrow p$

(27) $p \wedge \neg p \, \mathcal{U} \, q \Rightarrow q$

(28) $p \, \mathcal{U} \, q \Rightarrow p \vee q$

(29) $\mathcal{U}$ **insertion:** $q \Rightarrow p \, \mathcal{U} \, q$

(30) $p \wedge q \Rightarrow p \, \mathcal{U} \, q$

(31) **Absorption:** $p \vee p \, \mathcal{U} \, q \equiv p \vee q$

(32) **Absorption:** $p \, \mathcal{U} \, q \vee q \equiv p \, \mathcal{U} \, q$

(33) **Absorption:** $p \, \mathcal{U} \, q \wedge q \equiv q$

(34) **Absorption:** $p \, \mathcal{U} \, q \vee (p \wedge q) \equiv p \, \mathcal{U} \, q$

(35) **Absorption:** $p \, \mathcal{U} \, q \wedge (p \vee q) \equiv p \, \mathcal{U} \, q$

(36) **Left absorption of** $\mathcal{U}$ **:** $p \, \mathcal{U} \, (p \, \mathcal{U} \, q) \equiv p \, \mathcal{U} \, q$

(37) **Right absorption of** $\mathcal{U}$ **:** $(p \, \mathcal{U} \, q) \, \mathcal{U} \, q \equiv p \, \mathcal{U} \, q$

Eventually $\Diamond$

(38) **Definition of** $\Diamond$ **:** $\Diamond \, q \equiv true \, \mathcal{U} \, q$

(39) **Absorption of** $\Diamond$ **into** $\mathcal{U}$ **:** $p \, \mathcal{U} \, q \wedge \Diamond \, q \equiv p \, \mathcal{U} \, q$

(40) **Absorption of** $\mathcal{U}$ **into** $\Diamond$ **:** $p \, \mathcal{U} \, q \vee \Diamond \, q \equiv \Diamond \, q$

(41) **Absorption of** $\mathcal{U}$ **into** $\Diamond$ **:** $p \, \mathcal{U} \, \Diamond \, q \equiv \Diamond \, q$

(42) **Eventuality:** $p \, \mathcal{U} \, q \Rightarrow \Diamond \, q$

(43) **Truth of** $\Diamond$ **:** $\Diamond \, true \equiv true$

(44) **Falsehood of** $\Diamond$ **:** $\Diamond \, false \equiv false$

(45) **Expansion of** $\Diamond$ **:** $\Diamond \, p \equiv p \vee \bigcirc \Diamond \, p$

(46) **Weakening of** $\Diamond$ **:** $p \Rightarrow \Diamond \, p$

(47) **Weakening of** $\Diamond$ **:** $\bigcirc \, p \Rightarrow \Diamond \, p$

(48) **Absorption of** $\vee$ **into** $\Diamond$ **:** $p \vee \Diamond \, p \equiv \Diamond \, p$

(49) **Absorption of** $\Diamond$ **into** $\wedge$**:** $\Diamond \, p \wedge p \equiv p$

(50) **Absorption of** $\Diamond$ **:** $\Diamond \Diamond \, p \equiv \Diamond \, p$

(51) **Exchange of** $\bigcirc$ **and** $\Diamond$ **:** $\bigcirc \Diamond \, p \equiv \Diamond \bigcirc \, p$

(52) **Distributivity of** $\Diamond$ **over** $\vee$**:** $\Diamond \, (p \vee q) \equiv \Diamond \, p \vee \Diamond \, q$

(53) **Distributivity of** $\Diamond$ **over** $\wedge$**:** $\Diamond \, (p \wedge q) \Rightarrow \Diamond \, p \wedge \Diamond \, q$

Always $\Box$

(54) **Definition of $\Box$:** $\Box\, p \equiv \neg\Diamond\, \neg p$

(55) **Axiom, $\mathcal{U}$ Induction:** $\Box\,(p \Rightarrow (\bigcirc\, p \wedge q) \vee r) \Rightarrow (p \Rightarrow \Box\, q \vee q\, \mathcal{U}\, r)$

(56) **Axiom, $\mathcal{U}$ Induction:** $\Box\,(p \Rightarrow \bigcirc\,(p \vee q)) \Rightarrow (p \Rightarrow \Box\, p \vee p\, \mathcal{U}\, q)$

(57) **$\Box$ Induction:** $\Box\,(p \Rightarrow \bigcirc\, p) \Rightarrow (p \Rightarrow \Box\, p)$

(58) **$\Diamond$ Induction:** $\Box\,(\bigcirc\, p \Rightarrow p) \Rightarrow (\Diamond\, p \Rightarrow p)$

(59) $\Diamond\, p \equiv \neg\Box\, \neg p$

(60) **Dual of $\Box$:** $\neg\Box\, p \equiv \Diamond\, \neg p$

(61) **Dual of $\Diamond$:** $\neg\Diamond\, p \equiv \Box\, \neg p$

(62) **Dual of $\Diamond\, \Box$:** $\neg\Diamond\, \Box\, p \equiv \Box\, \Diamond\, \neg p$

(63) **Dual of $\Box\, \Diamond$:** $\neg\Box\, \Diamond\, p \equiv \Diamond\, \Box\, \neg p$

(64) **Truth of $\Box$:** $\Box\, true \equiv true$

(65) **Falsehood of $\Box$:** $\Box\, false \equiv false$

(66) **Expansion of $\Box$:** $\Box\, p \equiv p \wedge \bigcirc\, \Box\, p$

(67) **Expansion of $\Box$:** $\Box\, p \equiv p \wedge \bigcirc\, p \wedge \bigcirc\, \Box\, p$

(68) **Absorption of $\wedge$ into $\Box$:** $p \wedge \Box\, p \equiv \Box\, p$

(69) **Absorption of $\Box$ into $\vee$:** $\Box\, p \vee p \equiv p$

(70) **Absorption of $\Diamond$ into $\Box$:** $\Diamond\, p \wedge \Box\, p \equiv \Box\, p$

(71) **Absorption of $\Box$ into $\Diamond$:** $\Box\, p \vee \Diamond\, p \equiv \Diamond\, p$

(72) **Absorption of $\Box$:** $\Box\, \Box\, p \equiv \Box\, p$

(73) **Exchange of $\bigcirc$ and $\Box$:** $\bigcirc\, \Box\, p \equiv \Box\, \bigcirc\, p$

(74) $p \Rightarrow \Box\, p \equiv p \Rightarrow \bigcirc\, \Box\, p$

(75) $p \wedge \Diamond\, \neg p \Rightarrow \Diamond\,(p \wedge \bigcirc\, \neg p)$

(76) **Strengthening of $\Box$:** $\Box\, p \Rightarrow p$

(77) **Strengthening of $\Box$:** $\Box\, p \Rightarrow \Diamond\, p$

(78) **Strengthening of $\Box$:** $\Box\, p \Rightarrow \bigcirc\, p$

(79) **Strengthening of $\Box$:** $\Box\, p \Rightarrow \bigcirc\, \Box\, p$

(80) **$\bigcirc$ generalization:** $\Box\, p \Rightarrow \Box\, \bigcirc\, p$

(81) $\Box\, p \Rightarrow \neg(q\, \mathcal{U}\, \neg p)$

Temporal deduction

(82) **Temporal deduction:**

To prove $\Box P_1 \land \Box P_2 \Rightarrow Q$, assume P_1 and P_2, and prove Q.
You cannot use textual substitution in P_1 or P_2.

Always, continued

(83) **Distributivity of $\land$ over $\mathcal{U}$:** $\Box p \land q \, \mathcal{U} \, r \Rightarrow (p \land q) \, \mathcal{U} \, (p \land r)$

(84) **$\mathcal{U}$ implication:** $\Box p \land \Diamond q \Rightarrow p \, \mathcal{U} \, q$

(85) **Right monotonicity of $\mathcal{U}$:** $\Box (p \Rightarrow q) \Rightarrow (r \, \mathcal{U} \, p \Rightarrow r \, \mathcal{U} \, q)$

(86) **Left monotonicity of $\mathcal{U}$:** $\Box (p \Rightarrow q) \Rightarrow (p \, \mathcal{U} \, r \Rightarrow q \, \mathcal{U} \, r)$

(87) **Distributivity of $\neg$ over $\Box$:** $\Box \neg p \Rightarrow \neg \Box p$

(88) **Distributivity of $\Diamond$ over $\land$:** $\Box p \land \Diamond q \Rightarrow \Diamond (p \land q)$

(89) **$\Diamond$ excluded middle:** $\Diamond p \lor \Box \neg p$

(90) **$\Box$ excluded middle:** $\Box p \lor \Diamond \neg p$

(91) **Temporal excluded middle:** $\Diamond p \lor \Diamond \neg p$

(92) **$\Diamond$ contradiction:** $\Diamond p \land \Box \neg p \equiv false$

(93) **$\Box$ contradiction:** $\Box p \land \Diamond \neg p \equiv false$

(94) **Temporal contradiction:** $\Box p \land \Box \neg p \equiv false$

(95) **$\Box \Diamond$ excluded middle:** $\Box \Diamond p \lor \Diamond \Box \neg p$

(96) **$\Diamond \Box$ excluded middle:** $\Diamond \Box p \lor \Box \Diamond \neg p$

(97) **$\Box \Diamond$ contradiction:** $\Box \Diamond p \land \Diamond \Box \neg p \equiv false$

(98) **$\Diamond \Box$ contradiction:** $\Diamond \Box p \land \Box \Diamond \neg p \equiv false$

(99) **Distributivity of $\Box$ over $\land$:** $\Box (p \land q) \equiv \Box p \land \Box q$

(100) **Distributivity of $\Box$ over $\lor$:** $\Box p \lor \Box q \Rightarrow \Box (p \lor q)$

(101) **Logical equivalence law of $\bigcirc$:** $\Box (p \equiv q) \Rightarrow (\bigcirc p \equiv \bigcirc q)$

(102) **Logical equivalence law of $\Diamond$:** $\Box (p \equiv q) \Rightarrow (\Diamond p \equiv \Diamond q)$

(103) **Logical equivalence law of $\Box$:** $\Box (p \equiv q) \Rightarrow (\Box p \equiv \Box q)$

(104) **Distributivity of $\Diamond$ over $\Rightarrow$:** $\Diamond (p \Rightarrow q) \equiv (\Box p \Rightarrow \Diamond q)$

(105) **Distributivity of $\Diamond$ over $\Rightarrow$:** $(\Diamond p \Rightarrow \Diamond q) \Rightarrow \Diamond (p \Rightarrow q)$

(106) **$\land$ frame law of $\bigcirc$:** $\Box p \Rightarrow (\bigcirc q \Rightarrow \bigcirc (p \land q))$

(107) **$\land$ frame law of $\Diamond$:** $\Box p \Rightarrow (\Diamond q \Rightarrow \Diamond (p \land q))$

(108) **$\land$ frame law of $\Box$:** $\Box p \Rightarrow (\Box q \Rightarrow \Box (p \land q))$

(109) $\vee$ **frame law of** $\bigcirc$ **:** $\square\, p \Rightarrow (\bigcirc\, q \Rightarrow \bigcirc\,(p \vee q))$

(110) $\vee$ **frame law of** $\diamond$ **:** $\square\, p \Rightarrow (\diamond\, q \Rightarrow \diamond\,(p \vee q))$

(111) $\vee$ **frame law of** $\square$ **:** $\square\, p \Rightarrow (\square\, q \Rightarrow \square\,(p \vee q))$

(112) $\Rightarrow$ **frame law of** $\bigcirc$ **:** $\square\, p \Rightarrow (\bigcirc\, q \Rightarrow \bigcirc\,(p \Rightarrow q))$

(113) $\Rightarrow$ **frame law of** $\diamond$ **:** $\square\, p \Rightarrow (\diamond\, q \Rightarrow \diamond\,(p \Rightarrow q))$

(114) $\Rightarrow$ **frame law of** $\square$ **:** $\square\, p \Rightarrow (\square\, q \Rightarrow \square\,(p \Rightarrow q))$

(115) $\equiv$ **frame law of** $\bigcirc$ **:** $\square\, p \Rightarrow (\bigcirc\, q \Rightarrow \bigcirc\,(p \equiv q))$

(116) $\equiv$ **frame law of** $\diamond$ **:** $\square\, p \Rightarrow (\diamond\, q \Rightarrow \diamond\,(p \equiv q))$

(117) $\equiv$ **frame law of** $\square$ **:** $\square\, p \Rightarrow (\square\, q \Rightarrow \square\,(p \equiv q))$

(118) **Monotonicity of** $\bigcirc$ **:** $\square\,(p \Rightarrow q) \Rightarrow (\bigcirc\, p \Rightarrow \bigcirc\, q)$

(119) **Monotonicity of** $\diamond$ **:** $\square\,(p \Rightarrow q) \Rightarrow (\diamond\, p \Rightarrow \diamond\, q)$

(120) **Monotonicity of** $\square$ **:** $\square\,(p \Rightarrow q) \Rightarrow (\square\, p \Rightarrow \square\, q)$

(121) **Consequence rule of** $\bigcirc$ **:** $\square\,((p \Rightarrow q) \wedge (q \Rightarrow \bigcirc\, r) \wedge (r \Rightarrow s)) \Rightarrow (p \Rightarrow \bigcirc\, s)$

(122) **Consequence rule of** $\diamond$ **:** $\square\,((p \Rightarrow q) \wedge (q \Rightarrow \diamond\, r) \wedge (r \Rightarrow s)) \Rightarrow (p \Rightarrow \diamond\, s)$

(123) **Consequence rule of** $\square$ **:** $\square\,((p \Rightarrow q) \wedge (q \Rightarrow \square\, r) \wedge (r \Rightarrow s)) \Rightarrow (p \Rightarrow \square\, s)$

(124) **Catenation rule of** $\diamond$ **:** $\square\,((p \Rightarrow \diamond\, q) \wedge (q \Rightarrow \diamond\, r)) \Rightarrow (p \Rightarrow \diamond\, r)$

(125) **Catenation rule of** $\square$ **:** $\square\,((p \Rightarrow \square\, q) \wedge (q \Rightarrow \square\, r)) \Rightarrow (p \Rightarrow \square\, r)$

(126) **Catenation rule of** $\mathcal{U}$ **:** $\square\,((p \Rightarrow q\ \mathcal{U}\ r) \wedge (r \Rightarrow q\ \mathcal{U}\ s)) \Rightarrow (p \Rightarrow q\ \mathcal{U}\ s)$

(127) $\mathcal{U}$ **strengthening rule:** $\square\,((p \Rightarrow r) \wedge (q \Rightarrow s)) \Rightarrow (p\ \mathcal{U}\ q \Rightarrow r\ \mathcal{U}\ s)$

(128) **Induction rule** $\diamond$ **:** $\square\,(p \vee \bigcirc\, q \Rightarrow q) \Rightarrow (\diamond\, p \Rightarrow q)$

(129) **Induction rule** $\square$ **:** $\square\,(p \Rightarrow q \wedge \bigcirc\, p) \Rightarrow (p \Rightarrow \square\, q)$

(130) **Induction rule** $\mathcal{U}$ **:** $\square\,(p \Rightarrow \neg q \wedge \bigcirc\, p) \Rightarrow (p \Rightarrow \neg(r\ \mathcal{U}\ q))$

(131) $\diamond$ **Confluence:** $\square\,((p \Rightarrow \diamond\,(q \vee r)) \wedge (q \Rightarrow \diamond\, t) \wedge (r \Rightarrow \diamond\, t)) \Rightarrow (p \Rightarrow \diamond\, t)$

(132) **Temporal generalization law:** $\square\,(\square\, p \Rightarrow q) \Rightarrow (\square\, p \Rightarrow \square\, q)$

(133) **Temporal particularization law:** $\square\,(p \Rightarrow \diamond\, q) \Rightarrow (\diamond\, p \Rightarrow \diamond\, q)$

(134) $\square\,(p \Rightarrow \bigcirc\, q) \Rightarrow (p \Rightarrow \diamond\, q)$

(135) $\square\,(p \Rightarrow \bigcirc\, \neg p) \Rightarrow (p \Rightarrow \neg \square\, p)$

Proof metatheorems

(136) **Metatheorem:** P is a theorem iff $\Box P$ is a theorem.

(137) **Metatheorem $\bigcirc$:** If $P \Rightarrow Q$ is a theorem then $\bigcirc P \Rightarrow \bigcirc Q$ is a theorem.

(138) **Metatheorem $\Diamond$:** If $P \Rightarrow Q$ is a theorem then $\Diamond P \Rightarrow \Diamond Q$ is a theorem.

(139) **Metatheorem $\Box$:** If $P \Rightarrow Q$ is a theorem then $\Box P \Rightarrow \Box Q$ is a theorem.

Always, continued

(140) **$\mathcal{U}\,\Box$ implication:** $p\,\mathcal{U}\,\Box q \Rightarrow \Box(p\,\mathcal{U}\,q)$

(141) **Absorption of $\mathcal{U}$ into $\Box$:** $p\,\mathcal{U}\,\Box p \equiv \Box p$

(142) **Right $\wedge\,\mathcal{U}$ strengthening:** $p\,\mathcal{U}\,(q \wedge r) \Rightarrow p\,\mathcal{U}\,(q\,\mathcal{U}\,r)$

(143) **Left $\wedge\,\mathcal{U}$ strengthening:** $(p \wedge q)\,\mathcal{U}\,r \Rightarrow (p\,\mathcal{U}\,q)\,\mathcal{U}\,r$

(144) **Left $\wedge\,\mathcal{U}$ ordering:** $(p \wedge q)\,\mathcal{U}\,r \Rightarrow p\,\mathcal{U}\,(q\,\mathcal{U}\,r)$

(145) **$\Diamond\,\Box$ implication:** $\Diamond\Box p \Rightarrow \Box\Diamond p$

(146) **$\Box\,\Diamond$ excluded middle:** $\Box\Diamond p \vee \Box\Diamond\neg p$

(147) **$\Diamond\,\Box$ contradiction:** $\Diamond\Box p \wedge \Diamond\Box\neg p \;\equiv\; false$

(148) **$\mathcal{U}$ frame law of $\bigcirc$:** $\Box p \Rightarrow (\bigcirc q \Rightarrow \bigcirc(p\,\mathcal{U}\,q))$

(149) **$\mathcal{U}$ frame law of $\Diamond$:** $\Box p \Rightarrow (\Diamond q \Rightarrow \Diamond(p\,\mathcal{U}\,q))$

(150) **$\mathcal{U}$ frame law of $\Box$:** $\Box p \Rightarrow (\Box q \Rightarrow \Box(p\,\mathcal{U}\,q))$

(151) **Absorption of $\Diamond$ into $\Box\Diamond$:** $\Diamond\Box\Diamond p \equiv \Box\Diamond p$

(152) **Absorption of $\Box$ into $\Diamond\Box$:** $\Box\Diamond\Box p \equiv \Diamond\Box p$

(153) **Absorption of $\Box\Diamond$:** $\Box\Diamond\Box\Diamond p \equiv \Box\Diamond p$

(154) **Absorption of $\Diamond\Box$:** $\Diamond\Box\Diamond\Box p \equiv \Diamond\Box p$

(155) **Absorption of $\bigcirc$ into $\Box\Diamond$:** $\bigcirc\Box\Diamond p \equiv \Box\Diamond p$

(156) **Absorption of $\bigcirc$ into $\Diamond\Box$:** $\bigcirc\Diamond\Box p \equiv \Diamond\Box p$

(157) **Monotonicity of $\Box\Diamond$:** $\Box(p \Rightarrow q) \Rightarrow (\Box\Diamond p \Rightarrow \Box\Diamond q)$

(158) **Monotonicity of $\Diamond\Box$:** $\Box(p \Rightarrow q) \Rightarrow (\Diamond\Box p \Rightarrow \Diamond\Box q)$

(159) **Distributivity of $\Box\Diamond$ over $\wedge$:** $\Box\Diamond(p \wedge q) \Rightarrow \Box\Diamond p \wedge \Box\Diamond q$

(160) **Distributivity of $\Diamond\Box$ over $\vee$:** $\Diamond\Box p \vee \Diamond\Box q \Rightarrow \Diamond\Box(p \vee q)$

(161) **Distributivity of $\Box\Diamond$ over $\vee$:** $\Box\Diamond(p \vee q) \equiv \Box\Diamond p \vee \Box\Diamond q$

(162) **Distributivity of $\Diamond\Box$ over $\wedge$:** $\Diamond\Box(p \wedge q) \equiv \Diamond\Box p \wedge \Diamond\Box q$

(163) **Eventual latching:** $\Diamond\Box(p \Rightarrow \Box q) \equiv \Diamond\Box\neg p \vee \Diamond\Box q$

(164) $\Box(\Box\Diamond p \Rightarrow \Diamond q) \equiv \Diamond\Box\neg p \vee \Box\Diamond q$

(165) $\Box((p \vee \Box q) \wedge (\Box p \vee q)) \equiv \Box p \vee \Box q$

(166) $\Diamond\Box p \wedge \Box\Diamond q \Rightarrow \Box\Diamond(p \wedge q)$

(167) $\Box((\Box p \Rightarrow \Diamond q) \wedge (q \Rightarrow \bigcirc r)) \Rightarrow (\Box p \Rightarrow \bigcirc\Box\Diamond r)$

(168) **Progress proof rule:** $\Diamond\Box p \wedge \Box(\Box p \Rightarrow \Diamond q) \Rightarrow \Diamond q$

Wait $\mathcal{W}$

(169) **Definition of $\mathcal{W}$:** $p \,\mathcal{W}\, q \equiv \Box\, p \vee p \,\mathcal{U}\, q$

(170) **Axiom, Distributivity of $\neg$ over $\mathcal{W}$:** $\neg(p \,\mathcal{W}\, q) \equiv \neg q \,\mathcal{U}\, (\neg p \wedge \neg q)$

(171) **$\mathcal{U}$ in terms of $\mathcal{W}$:** $p \,\mathcal{U}\, q \equiv p \,\mathcal{W}\, q \wedge \Diamond\, q$

(172) $p \,\mathcal{W}\, q \equiv \Box\, (p \wedge \neg q) \vee p \,\mathcal{U}\, q$

(173) **Distributivity of $\neg$ over $\mathcal{U}$:** $\neg(p \,\mathcal{U}\, q) \equiv \neg q \,\mathcal{W}\, (\neg p \wedge \neg q)$

(174) **$\mathcal{U}$ implication:** $p \,\mathcal{U}\, q \Rightarrow p \,\mathcal{W}\, q$

(175) **Distributivity of $\wedge$ over $\mathcal{W}$:** $\Box\, p \wedge q \,\mathcal{W}\, r \Rightarrow (p \wedge q) \,\mathcal{W}\, (p \wedge r)$

(176) **$\mathcal{W}\, \Diamond$ equivalence:** $p \,\mathcal{W}\, \Diamond\, q \equiv \Box\, p \vee \Diamond\, q$

(177) **$\mathcal{W}\, \Box$ implication:** $p \,\mathcal{W}\, \Box\, q \Rightarrow \Box\, (p \,\mathcal{W}\, q)$

(178) **Absorption of $\mathcal{W}$ into $\Box$:** $p \,\mathcal{W}\, \Box\, p \equiv \Box\, p$

(179) **Perpetuity:** $\Box\, p \Rightarrow p \,\mathcal{W}\, q$

(180) **Distributivity of $\bigcirc$ over $\mathcal{W}$:** $\bigcirc\, (p \,\mathcal{W}\, q) \equiv \bigcirc\, p \,\mathcal{W}\, \bigcirc\, q$

(181) **Expansion of $\mathcal{W}$:** $p \,\mathcal{W}\, q \equiv q \vee (p \wedge \bigcirc\, (p \,\mathcal{W}\, q))$

(182) **$\mathcal{W}$ excluded middle:** $p \,\mathcal{W}\, q \vee p \,\mathcal{W}\, \neg q$

(183) **Left zero of $\mathcal{W}$:** $true \,\mathcal{W}\, q \equiv true$

(184) **Left distributivity of $\mathcal{W}$ over $\vee$:** $p \,\mathcal{W}\, (q \vee r) \equiv p \,\mathcal{W}\, q \vee p \,\mathcal{W}\, r$

(185) **Right distributivity of $\mathcal{W}$ over $\vee$:** $p \,\mathcal{W}\, r \vee q \,\mathcal{W}\, r \Rightarrow (p \vee q) \,\mathcal{W}\, r$

(186) **Left distributivity of $\mathcal{W}$ over $\wedge$:** $p \,\mathcal{W}\, (q \wedge r) \Rightarrow p \,\mathcal{W}\, q \wedge p \,\mathcal{W}\, r$

(187) **Right distributivity of $\mathcal{W}$ over $\wedge$:** $(p \wedge q) \,\mathcal{W}\, r \equiv p \,\mathcal{W}\, r \wedge q \,\mathcal{W}\, r$

(188) **Right distributivity of $\mathcal{W}$ over $\Rightarrow$:** $(p \Rightarrow q) \,\mathcal{W}\, r \Rightarrow (p \,\mathcal{W}\, r \Rightarrow q \,\mathcal{W}\, r)$

(189) **Disjunction rule of $\mathcal{W}$:** $p \,\mathcal{W}\, q \equiv (p \vee q) \,\mathcal{W}\, q$

(190) **Disjunction rule of $\mathcal{U}$:** $p \,\mathcal{U}\, q \equiv (p \vee q) \,\mathcal{U}\, q$

(191) **Rule of $\mathcal{W}$:** $\neg q \,\mathcal{W}\, q$

(192) **Rule of $\mathcal{U}$:** $\neg q \,\mathcal{U}\, q \;\equiv\; \Diamond\, q$

(193) $(p \Rightarrow q) \,\mathcal{W}\, p$

(194) $\Diamond\, p \Rightarrow (p \Rightarrow q) \,\mathcal{U}\, p$

(195) **Conjunction rule of $\mathcal{W}$:** $p \mathcal{W} q \equiv (p \wedge \neg q) \mathcal{W} q$

(196) **Conjunction rule of $\mathcal{U}$:** $p \mathcal{U} q \equiv (p \wedge \neg q) \mathcal{U} q$

(197) **Distributivity of $\neg$ over $\mathcal{W}$:** $\neg(p \mathcal{W} q) \equiv (p \wedge \neg q) \mathcal{U} (\neg p \wedge \neg q)$

(198) **Distributivity of $\neg$ over $\mathcal{U}$:** $\neg(p \mathcal{U} q) \equiv (p \wedge \neg q) \mathcal{W} (\neg p \wedge \neg q)$

(199) **Dual of $\mathcal{U}$:** $\neg(\neg p \mathcal{U} \neg q) \equiv q \mathcal{W} (p \wedge q)$

(200) **Dual of $\mathcal{U}$:** $\neg(\neg p \mathcal{U} \neg q) \equiv (\neg p \wedge q) \mathcal{W} (p \wedge q)$

(201) **Dual of $\mathcal{W}$:** $\neg(\neg p \mathcal{W} \neg q) \equiv q \mathcal{U} (p \wedge q)$

(202) **Dual of $\mathcal{W}$:** $\neg(\neg p \mathcal{W} \neg q) \equiv (\neg p \wedge q) \mathcal{U} (p \wedge q)$

(203) **Idempotency of $\mathcal{W}$:** $p \mathcal{W} p \equiv p$

(204) **Right zero of $\mathcal{W}$:** $p \mathcal{W} true \equiv true$

(205) **Left identity of $\mathcal{W}$:** $false \mathcal{W} q \equiv q$

(206) $p \mathcal{W} q \Rightarrow p \vee q$

(207) $\Box (p \vee q) \Rightarrow p \mathcal{W} q$

(208) $\Box (\neg q \Rightarrow p) \Rightarrow p \mathcal{W} q$

(209) **$\mathcal{W}$ insertion:** $q \Rightarrow p \mathcal{W} q$

(210) **$\mathcal{W}$ frame law of $\bigcirc$:** $\Box p \Rightarrow (\bigcirc q \Rightarrow \bigcirc (p \mathcal{W} q))$

(211) **$\mathcal{W}$ frame law of $\Diamond$:** $\Box p \Rightarrow (\Diamond q \Rightarrow \Diamond (p \mathcal{W} q))$

(212) **$\mathcal{W}$ frame law of $\Box$:** $\Box p \Rightarrow (\Box q \Rightarrow \Box (p \mathcal{W} q))$

(213) **$\mathcal{W}$ induction:** $\Box (p \Rightarrow (\bigcirc p \wedge q) \vee r) \Rightarrow (p \Rightarrow q \mathcal{W} r)$

(214) **$\mathcal{W}$ induction:** $\Box (p \Rightarrow \bigcirc (p \vee q)) \Rightarrow (p \Rightarrow p \mathcal{W} q)$

(215) **$\mathcal{W}$ induction:** $\Box (p \Rightarrow \bigcirc p) \Rightarrow (p \Rightarrow p \mathcal{W} q)$

(216) **$\mathcal{W}$ induction:** $\Box (p \Rightarrow q \wedge \bigcirc p) \Rightarrow (p \Rightarrow p \mathcal{W} q)$

(217) **Absorption:** $p \vee p \mathcal{W} q \equiv p \vee q$

(218) **Absorption:** $p \mathcal{W} q \vee q \equiv p \mathcal{W} q$

(219) **Absorption:** $p \mathcal{W} q \wedge q \equiv q$

(220) **Absorption:** $p \mathcal{W} q \wedge (p \vee q) \equiv p \mathcal{W} q$

(221) **Absorption:** $p \mathcal{W} q \vee (p \wedge q) \equiv p \mathcal{W} q$

(222) **Left absorption of $\mathcal{W}$:** $p \mathcal{W} (p \mathcal{W} q) \equiv p \mathcal{W} q$

(223) **Right absorption of $\mathcal{W}$:** $(p \mathcal{W} q) \mathcal{W} q \equiv p \mathcal{W} q$

(224) **$\Box$ to $\mathcal{W}$ law:** $\Box p \equiv p \mathcal{W} false$

(225) **$\Diamond$ to $\mathcal{W}$ law:** $\Diamond p \equiv \neg(\neg p \mathcal{W} false)$

(226) **$\mathcal{W}$ implication:** $p \mathcal{W} q \Rightarrow \Box p \vee \Diamond q$

(227) **Absorption:** $p \,\mathcal{W}\, (p \,\mathcal{U}\, q) \equiv p \,\mathcal{W}\, q$

(228) **Absorption:** $(p \,\mathcal{U}\, q) \,\mathcal{W}\, q \equiv p \,\mathcal{U}\, q$

(229) **Absorption:** $p \,\mathcal{U}\, (p \,\mathcal{W}\, q) \equiv p \,\mathcal{W}\, q$

(230) **Absorption:** $(p \,\mathcal{W}\, q) \,\mathcal{U}\, q \equiv p \,\mathcal{U}\, q$

(231) **Absorption of $\mathcal{W}$ into $\Diamond$:** $\Diamond q \,\mathcal{W}\, q \equiv \Diamond q$

(232) **Absorption of $\mathcal{W}$ into $\Box$:** $\Box p \wedge p \,\mathcal{W}\, q \equiv \Box p$

(233) **Absorption of $\Box$ into $\mathcal{W}$:** $\Box p \vee p \,\mathcal{W}\, q \equiv p \,\mathcal{W}\, q$

(234) $p \,\mathcal{W}\, q \wedge \Box \neg q \Rightarrow \Box p$

(235) $\Box p \Rightarrow p \,\mathcal{U}\, q \vee \Box \neg q$

(236) $\neg \Box p \wedge p \,\mathcal{W}\, q \Rightarrow \Diamond q$

(237) $\Diamond q \Rightarrow \neg \Box p \vee p \,\mathcal{U}\, q$

(238) **Left monotonicity of $\mathcal{W}$:** $\Box (p \Rightarrow q) \Rightarrow (p \,\mathcal{W}\, r \Rightarrow q \,\mathcal{W}\, r)$

(239) **Right monotonicity of $\mathcal{W}$:** $\Box (p \Rightarrow q) \Rightarrow (r \,\mathcal{W}\, p \Rightarrow r \,\mathcal{W}\, q)$

(240) **$\mathcal{W}$ strengthening rule:** $\Box ((p \Rightarrow r) \wedge (q \Rightarrow s)) \Rightarrow (p \,\mathcal{W}\, q \Rightarrow r \,\mathcal{W}\, s)$

(241) **$\mathcal{W}$ catenation rule:** $\Box ((p \Rightarrow q \,\mathcal{W}\, r) \wedge (r \Rightarrow q \,\mathcal{W}\, s)) \Rightarrow (p \Rightarrow q \,\mathcal{W}\, s)$

(242) **Left $\mathcal{U}$ $\mathcal{W}$ implication:** $(p \,\mathcal{U}\, q) \,\mathcal{W}\, r \Rightarrow (p \,\mathcal{W}\, q) \,\mathcal{W}\, r$

(243) **Right $\mathcal{W}$ $\mathcal{U}$ implication:** $p \,\mathcal{W}\, (q \,\mathcal{U}\, r) \Rightarrow p \,\mathcal{W}\, (q \,\mathcal{W}\, r)$

(244) **Right $\mathcal{U}$ $\mathcal{U}$ implication:** $p \,\mathcal{U}\, (q \,\mathcal{U}\, r) \Rightarrow p \,\mathcal{U}\, (q \,\mathcal{W}\, r)$

(245) **Left $\mathcal{U}$ $\mathcal{U}$ implication:** $(p \,\mathcal{U}\, q) \,\mathcal{U}\, r \Rightarrow (p \,\mathcal{W}\, q) \,\mathcal{U}\, r$

(246) **Left $\mathcal{U}$ $\vee$ strengthening:** $(p \,\mathcal{U}\, q) \,\mathcal{U}\, r \Rightarrow (p \vee q) \,\mathcal{U}\, r$

(247) **Left $\mathcal{W}$ $\vee$ strengthening:** $(p \,\mathcal{W}\, q) \,\mathcal{W}\, r \Rightarrow (p \vee q) \,\mathcal{W}\, r$

(248) **Right $\mathcal{W}$ $\vee$ strengthening:** $p \,\mathcal{W}\, (q \,\mathcal{W}\, r) \Rightarrow p \,\mathcal{W}\, (q \vee r)$

(249) **Right $\mathcal{W}$ $\vee$ ordering:** $p \,\mathcal{W}\, (q \,\mathcal{W}\, r) \Rightarrow (p \vee q) \,\mathcal{W}\, r$

(250) **Right $\wedge$ $\mathcal{W}$ ordering:** $p \,\mathcal{W}\, (q \wedge r) \Rightarrow (p \,\mathcal{W}\, q) \,\mathcal{W}\, r$

(251) **$\mathcal{U}$ ordering:** $\neg p \,\mathcal{U}\, q \vee \neg q \,\mathcal{U}\, p \equiv \Diamond (p \vee q)$

(252) **$\mathcal{W}$ ordering:** $\neg p \,\mathcal{W}\, q \vee \neg q \,\mathcal{W}\, p$

(253) **$\mathcal{W}$ implication ordering:** $p \,\mathcal{W}\, q \wedge \neg q \,\mathcal{W}\, r \Rightarrow p \,\mathcal{W}\, r$

(254) **Lemmon formula:** $\Box (\Box p \Rightarrow q) \vee \Box (\Box q \Rightarrow p)$