Boolean operators

Operator: Conjunction

Symbol: $\wedge$

English: p and q

Conjuncts: p, q

Truth table:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Operator: Disjunction

Symbol: $\vee$

English: p or q

Disjuncts: p, q

Truth table:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Boolean operators

Operator: Implication

Symbol: $\Rightarrow$

English: p implies q

English: if p then q

Antecedent: p

Consequent: q

Truth table:

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Operator: Implication

Symbol: $\Leftarrow$

English: p follows from q

English: if q then p

Antecedent: q

Consequent: p

Truth table:

p	q	$p \Leftarrow q$
T	T	T
T	F	T
F	T	F
F	F	T

Boolean operators

Operator: Equivalence
Symbol: $\equiv$
English: p equivales q
English: p equals q
Truth table:

p	q	$p \equiv q$
T	T	T
T	F	F
F	T	F
F	F	T

Operator: Inequivalence
Symbol: $\not\equiv$
English: p exclusive or q
English: p different from q
Truth table:

p	q	$p \not\equiv q$
T	T	F
T	F	T
F	T	T
F	F	F

Boolean operators

Operator: Negation

Symbol: $\neg$

English: not p

Truth table:

p	$\neg p$
T	F
F	T

A Logical Approach to Discrete Math

p	q	r	$\neg r$	$q \wedge \neg r$	$p \vee (q \wedge \neg r)$
t	t	t	f	f	t
t	t	f	t	t	t
t	f	t	f	f	t
t	f	f	t	f	t
f	t	t	f	f	f
f	t	f	t	t	t
f	f	t	f	f	f
f	f	f	t	f	f

TABLE OF PRECEDENCES

(a) $[x := e]$ (textual substitution) (highest precedence)
(b) . (function application)
(c) unary prefix operators: $+ \quad - \quad \neg \quad \# \quad \sim \quad \mathcal{P}$
(d) $**$
(e) $\cdot \quad / \quad \div$ **mod** **gcd**
(f) $+ \quad - \quad \cup \quad \cap \quad \times \quad \circ \quad \bullet$
(g) $\downarrow \quad \uparrow$
(h) $\#$
(i) $\lhd \quad \rhd \quad \hat{}$
(j) $= \quad < \quad > \quad \in \quad \subset \quad \subseteq \quad \supset \quad \supseteq \quad |$ (conjunctional, see page 29)
(k) $\vee \quad \wedge$
(l) $\Rightarrow \quad \Leftarrow$
(m) $\equiv$

All nonassociative binary infix operators associate from left to right except $**$, $\lhd$, and $\Rightarrow$, which associate from right to left.

Definition of $/$: The operators on lines (j), (l), and (m) may have a slash $/$ through them to denote negation—e.g. $x \notin T$ is an abbreviation for $\neg(x \in T)$.

Precedence

$\wedge$ and $\vee$ have the same precedence.

When they are together, parentheses are necessary.

$p \wedge q \vee r$ Incorrect

Must be written $(p \wedge q) \vee r$ or $p \wedge (q \vee r)$

Equality vs Equivalence

$=$ and $\equiv$ have the same truth tables.

But, there are four differences:

(1) $=$ is for numbers and booleans.

$\equiv$ is only for booleans.

(2) $=$ has higher precedence than $\equiv$

Example: Do not need parentheses for

$$x \cdot y = 0 \quad \equiv \quad x = 0 \lor y = 0$$

Equality vs Equivalence

(3) $=$ is conjunctional, while $\equiv$ is not.

$$b = c = d \quad \text{means} \quad b = c \wedge c = d.$$

(4) $\equiv$ is associative, while $=$ is not.

$$(b \equiv c) \equiv d \quad \text{is equivalent to} \quad b \equiv (c \equiv d)$$

Equality vs Equivalence

Example

Consider the state $(b, false), (c, false).(d, true)$

$(b \equiv c) \equiv d$	$b \equiv (c \equiv d)$	$b = c = d$
$(F \equiv F) \equiv T$	$F \equiv (F \equiv T)$	$b = c \wedge c = d$
$T \equiv T$	$F \equiv F$	$F = F \wedge F = T$
T	T	$T \wedge F$
	Same	F

A Logical Approach to Discrete Math

(2.1) **Definition.** A boolean expression P is *satisfied* in a state if its value is $true$ in that state; P is *satisfiable* if there is a state in which it is satisfied; and P is *valid* if it is satisfied in every state. A valid boolean expression is called a *tautology*.

Example

Is $p \wedge \neg q$ satisfied in state $(p, true), (q, true)$?	no
Is $p \wedge \neg q$ satisfiable?	yes
Is $p \wedge \neg q$ valid?	no
Is $p \wedge \neg q$ a tautology?	no

A Logical Approach to Discrete Math

(2.1) **Definition.** A boolean expression P is *satisfied* in a state if its value is *true* in that state; P is *satisfiable* if there is a state in which it is satisfied; and P is *valid* if it is satisfied in every state. A valid boolean expression is called a *tautology*.

Example

p	q	$p \Rightarrow q$	$p \wedge \neg q$	$\neg(p \wedge \neg q)$	$p \Rightarrow q \equiv \neg(p \wedge \neg q)$
T	T	T	F	T	T
T	F	F	T	F	T
F	T	T	F	T	T
F	F	T	F	T	T

So, $p \Rightarrow q \equiv \neg(p \wedge \neg q)$ is valid, i.e. is a tautology.

A Logical Approach to Discrete Math

(2.2) **Definition.** The *dual* P_D of a boolean expression P is constructed from P by interchanging occurrences of

$$\begin{array}{l} \mathit{true} \ \text{ and } \ \mathit{false}\,, \\ \wedge \ \text{ and } \ \vee\,, \\ \equiv \ \text{ and } \ \not\equiv\,, \\ \Rightarrow \ \text{ and } \ \not\Leftarrow\,, \text{ and} \\ \Leftarrow \ \text{ and } \ \not\Rightarrow\,. \end{array}$$

TABLE 2.1. EXAMPLES OF DUALS

P	P_D
$p \vee q$	$p \wedge q$
$p \Rightarrow q$	$p \not\Leftarrow q$
$p \equiv \neg p$	$p \not\equiv \neg p$
$\mathit{false} \not\equiv \mathit{true} \vee p$	$\mathit{true} \equiv \mathit{false} \wedge p$
$\neg p \wedge \neg q \equiv r$	$\neg p \vee \neg q \not\equiv r$

A Logical Approach to Discrete Math

(2.3) **Metatheorem Duality:**

(a) $valid(P) \;\equiv\; valid(\neg P_D)$

(b) $valid(P \equiv Q) \;\equiv\; valid(P_D \equiv Q_D)$

TABLE 2.2. USING DUALITY TO GENERATE VALID EXPRESSIONS

P (valid)	$\neg P_D$ (also valid)
$true$	$\neg false$
$p \vee true$	$\neg(p \wedge false)$
$p \vee \neg p$	$\neg(p \wedge \neg p)$

$P \equiv Q$ (valid)	$P_D \equiv Q_D$ (also valid)
$true \equiv true$	$false \equiv false$
$p \vee q \equiv q \vee p$	$p \wedge q \equiv q \wedge p$
$p \equiv q \equiv q \equiv p$	$p \not\equiv q \equiv q \not\equiv p$
$\neg(p \vee q) \equiv \neg p \wedge \neg q$	$\neg(p \wedge q) \equiv \neg p \vee \neg q$

(2.5) **Translation into a boolean expression.** To translate proposition p into a boolean expression:

1. Introduce boolean variables to denote subpropositions.
2. Replace these subpropositions by their corresponding boolean variables.
3. Translate the result of step 2 into a boolean expression, using "obvious" translations of the English words into operators. Table 2.3 gives examples of translations of English words.

TABLE 2.3. TRANSLATION OF ENGLISH WORDS

and	becomes	$\wedge$
or	becomes	$\vee$
not	becomes	$\neg$
it is not the case that	becomes	$\neg$
if p then q	becomes	$p \Rightarrow q$

A Logical Approach to Discrete Math

x : Henry VIII had one son,

y : Cleopatra had two (sons),

z : I'll eat my hat,

w : 1 is prime.

We then have the following sentences and their translations.

proposition	translation
Henry VIII had one son or I'll eat my hat.	$x \vee z$.
Henry VIII had one son and 1 is not prime.	$x \wedge \neg w$.
If 1 is prime and Cleopatra had two sons, I'll eat my hat.	$w \wedge y \Rightarrow z$.

English expression 1. p, if q

Same as, If q then p, $q \Rightarrow p$

English expression 2. p, only if q

Same as, If p then q, $p \Rightarrow q$

Be careful. This is *not* the same as English expression 1.

For example,

"You can be president, only if you are at least 35 years old."

means

"If you are president, then you are at least 35 years old."

English expression 3. p, if and only if q

Same as, $(q \Rightarrow p) \wedge (p \Rightarrow q)$

Same as, $p \equiv q$

Abbreviated in English as p iff q

English expression 4. p is a sufficient condition for q

Same as, If p then q, $p \Rightarrow q$

English expression 5. p is a necessary condition for q

Same as, If q then p, $q \Rightarrow p$

Remember 4. and 5. by sufficient $\Rightarrow$ necessary

English expression 6. p is a necessary and sufficient condition for q

Same as, $p \equiv q$

English expression 7. p, whenever q

Same as, If q then p, $q \Rightarrow p$

"Whenever" means the same thing as "if".

English expression 8. p, provided that q

Same as, If q then p, $q \Rightarrow p$

"Provided that" means the same thing as "if".

English expression 9. p, unless q

Same as, If not q then p, $\neg q \Rightarrow p$

For example,

"I will buy it, unless you do."

means

"If you do not buy it, then I will buy it."

English expression 10. p, unless not q

Same as, If q then p, $q \Rightarrow p$

For example,

"I will take another course, unless I do not pass this one."

means

"If I pass this course, then I'll take another."

English expression 11. $p\ ;\ q$

Same as, p and q, $p \wedge q$

English expression 12. $p\ ,\ q$

Same as, p and q, $p \wedge q$

English expression 13. p, but q

Same as, p and q, $p \wedge q$

For example, “I can do that, but so can you.”

means

“I can do that, and you can do that.”