A Logical Approach to Discrete Math

(5.1) If Joe fails to submit a project in course CS414, then he fails the course. If Joe fails CS414, then he cannot graduate. Hence, if Joe graduates, he must have submitted a project.

s : Joe submits a project in CS414.
f : Joe fails CS414.
g : Joe graduates.

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(5.2) If X is greater than zero, then if Y is zero then Z is zero. Variable Y is zero. Hence, either X is greater than zero or Z is zero.

This argument consists of two facts and a conclusion drawn from them. We begin formalizing the argument by associating identifiers with its primitive propositions.

x : X is greater than zero.
y : Y is zero.
z : Z is zero.

We can then formalize (5.2) as

(5.3) $(x \Rightarrow (y \Rightarrow z)) \wedge y \Rightarrow x \vee z$.

TABLE 5.1. COUNTEREXAMPLES FOR EXPRESSIONS

expression	counterexample 1	counterexample 2
$p \wedge q$	$p = \mathit{false}$	$q = \mathit{false}$
$p \vee q$	$p = q = \mathit{false}$	
$p \equiv q$	$p = \mathit{true}\ ,\ \ q = \mathit{false}$	$p = \mathit{false}\ ,\ \ q = \mathit{true}$
$p \not\equiv q$	$p = q = \mathit{true}$	$p = q = \mathit{false}$
$p \Rightarrow q$	$p = \mathit{true}\ ,\ \ q = \mathit{false}$	

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Consider the following, which is a simplification of a situation in Shakespeare's *Merchant of Venice*. Portia has a gold casket and a silver casket and has placed a picture of herself in one of them. On the caskets, she has written the following inscriptions:

Gold: The portrait is not in here.
Silver: Exactly one of these inscriptions is true.

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Portia explains to her suitor that each inscription may be *true* or *false*, but that she has placed her portrait in one of the caskets in a manner that is consistent with this truth or falsity of the inscriptions. If he can choose the casket with her portrait, she will marry him —in those days, that's what suitors wanted. The problem for the suitor is to use the inscriptions (although they could be *true* or *false*) to determine which casket contains her portrait.

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To begin solving the problem, we formalize it. We introduce four variables to stand for primitive propositions:

gc : The portrait is in the gold casket.
sc : The portrait is in the silver casket.
g : The portrait is not in the gold casket.
(This the inscription on the gold casket.)
s : Exactly one of g and s is *true*.
(This the inscription on the silver casket.)

> If Superman were able and willing to prevent evil, he would do so. If Superman were unable to prevent evil, he would be impotent; if he were unwilling to prevent evil, he would be malevolent. Superman does not prevent evil. If Superman exists, he is neither impotent nor malevolent. Therefore, Superman does not exist.

We want to use the propositional calculus to determine whether this argument is sound —whether the conclusion "Superman does not exist" follows from the previous sentences. As on page 37, we associate variables with the primitive propositions:

a : Superman is able to prevent evil.
w : Superman is willing to prevent evil.
i : Superman is impotent.
m : Superman is malevolent.
p : Superman prevents evil.
e : Superman exists.

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5.6 This set of questions concerns an island of knights and knaves. Knights always tell the truth and knaves always lie. In formalizing these questions, associate identifiers as follows:

b : B is a knight.
c : C is a knight.
d : D is a knight.

If B says a statement "X", this gives rise to the expression $b \equiv X$, since if b, then B is a knight and tells the truth, and if $\neg b$, B is a knave and lies.

(a) Someone asks B "are you a knight?" He replies, "If I am a knight, I'll eat my hat." Prove that B has to eat his hat.

(b) Inhabitant B says of inhabitant C, "If C is a knight, then I am a knave." What are B and C?

(c) It is rumored that gold is buried on the island. You ask B whether there is gold on the island. He replies, "There is gold on the island if and only if I am a knight." Can it be determined whether B is a knight or a knave? Can it be determined whether there is gold on the island?

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(d) Three inhabitants are standing together in the garden. A non-inhabitant passes by and asks B, "Are you a knight or a knave?" B answers, but so indistinctly that the stranger cannot understand. The stranger then asks C, "What did B say?" C replies, "B said that he is a knave." At this point, the third man, D, says, "Don't believe C; he's lying!" What are C and D?
Hint: Only C's and D's statements are relevant to the problem. Also, D's remark that C is lying is equivalent to saying that C is a knave.