For symmetric and associative binary operator $\star$ with identity u.

(8.13) **Axiom, Empty range:** $(\star x \mid false : P) = u$

0 is the identity of $+$.

$(+i \mid 2 \le i < 5 : i^2) = 2^2 + 3^2 + 4^2$

$(+i \mid 2 \le i < 4 : i^2) = 2^2 + 3^2$

$(+i \mid 2 \le i < 3 : i^2) = 2^2$

$(+i \mid 2 \le i < 2 : i^2) = (+i \mid false : i^2) = 0$

$true$ is the identity of $\wedge$.

Suppose b is an array of integers.

$(\wedge i \mid 2 \le i < 5 : b[i] < x) = b[2] < x \wedge b[3] < x \wedge b[4] < x$

$(\wedge i \mid 2 \le i < 4 : b[i] < x) = b[2] < x \wedge b[3] < x$

$(\wedge i \mid 2 \le i < 3 : b[i] < x) = b[2] < x$

$(\wedge i \mid 2 \le i < 2 : b[i] < x) = (+x \mid false : b[i] < x) = true$

(8.14) **Axiom, One-point rule:** Provided $\neg occurs(`x', `E')$,

$(\star x \mid x = E : P) = P[x := E]$

$(+i \mid i = 3 : i^2) \ = \ i^2\ [i := 3] \ = \ 3^2$

Suppose b is an array of integers.

$(\vee i \mid i = 3 : b[i] < x) \ = \ (b[i] < x)\ [i := 3] \ = \ b[3] < x$

(8.15) **Axiom, Distributivity:** Provided $P, Q : \mathbb{B}$ or R is finite,

$(\star x \mid R : P) \star (\star x \mid R : Q) = (\star x \mid R : P \star Q)$

$$
\begin{aligned}
& (+i \mid 1 \le i < 4 : 2i) + (+i \mid 1 \le i < 4 : 5i^2) \\
= \quad & \langle \text{Expand quantifications} \rangle \\
& (2 \cdot 1 + 2 \cdot 2 + 2 \cdot 3) + (5 \cdot 1^2 + 5 \cdot 2^2 + 5 \cdot 3^2) \\
= \quad & \langle \text{Symmetry and associativity of } + \rangle \\
& (2 \cdot 1 + 5 \cdot 1^2) + (2 \cdot 2 + 5 \cdot 2^2) + (2 \cdot 3 + 5 \cdot 3^2) \\
= \quad & \langle \text{Quantify} \rangle \\
& (+i \mid 1 \le i < 4 : 2i + 5i^2)
\end{aligned}
$$

(8.16) **Axiom, Range split:** Provided $R \wedge S \equiv false$ and $P : \mathbb{B}$ or R and S are finite,
$(\star x \mid R \vee S : P) = (\star x \mid R : P) \star (\star x \mid S : P)$

$R :\ 0 \leq i < 3$
$S :\ 6 \leq i < 9$
$R \vee S :\ 0 \leq i < 3 \ \vee\ 6 \leq i < 9$
$R \wedge S :\ false$

$(+i \mid R \vee S : i^2)$
$=$ ⟨Definition of R and S⟩
$(+i \mid 0 \leq i < 3 \ \vee\ 6 \leq i < 9 : i^2)$
$=$ ⟨Expand quantification⟩
$0^2 + 1^2 + 2^2 + 6^2 + 7^2 + 8^2$
$=$ ⟨Associativity of $+$⟩
$(0^2 + 1^2 + 2^2) + (6^2 + 7^2 + 8^2)$
$=$ ⟨Quantify⟩
$(+i \mid 0 \leq i < 3 : i^2) + (+i \mid 6 \leq i < 9 : i^2)$
$=$ ⟨Definition of R and S⟩
$(+i \mid R : i^2) + (+i \mid S : i^2)$

(8.17) **Axiom, Range split:** Provided $P : \mathbb{B}$ or R and S are finite,
$(\star x \mid R \vee S : P) \star (\star x \mid R \wedge S : P) = (\star x \mid R : P) \star (\star x \mid S : P)$

Now, $R \wedge S$ is not required to be false.

$R :\ 1 \leq i < 5$
$S :\ 3 \leq i < 7$
$R \vee S :\ 1 \leq i < 7$
$R \wedge S :\ 3 \leq i < 5$

$(+i \mid R \vee S : i^2) + (+i \mid R \wedge S : i^2)$
$=$ ⟨Definition of R and S⟩
$(+i \mid 1 \leq i < 7 : i^2) + (+i \mid 3 \leq i < 5 : i^2)$
$=$ ⟨Expand quantifications⟩
$(1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2) + (3^2 + 4^2)$
$=$ ⟨Symmetry and associativity of $+$⟩
$(1^2 + 2^2 + 3^2 + 4^2) + (3^2 + 4^2 + 5^2 + 6^2)$
$=$ ⟨Quantify⟩
$(+i \mid 1 \leq i < 5 : i^2) + (+i \mid 4 \leq i < 5 : i^2)$
$=$ ⟨Definition of R and S⟩
$(+i \mid R : i^2) + (+i \mid S : i^2)$

(8.18) **Axiom, Range split for idempotent $\star$:** Provided $P:\mathbb{B}$ or R and S are finite,
$(\star x \mid R \vee S : P) = (\star x \mid R : P) \star (\star x \mid S : P)$

$\wedge$ is idempotent because $p \wedge p \equiv p$.
Suppose b is an array of integers.
$R:\ 0 \le i < 2$
$S:\ 1 \le i < 3$
$R \vee S:\ 0 \le i < 3$

$(\wedge i \mid R : x < b[i]) \ \wedge\ (\wedge i \mid S : x < b[i])$
$=$ $\langle$Definition of R and $S\rangle$
$(\wedge i \mid 0 \le i < 2 : x < b[i]) \ \wedge\ (\wedge i \mid 1 \le i < 3 : x < b[i])$
$=$ $\langle$Expand quantifications$\rangle$
$x < b[0] \ \wedge\ x < b[1] \ \wedge\ x < b[1] \ \wedge\ x < b[2]$
$=$ $\langle$(3.38) $p \wedge p \equiv p\rangle$
$x < b[0] \ \wedge\ x < b[1] \ \wedge\ x < b[2]$
$=$ $\langle$Quantify$\rangle$
$(\wedge i \mid 0 \le i < 3 : x < b[i])$
$=$ $\langle$Definition of R and $S\rangle$
$(\wedge i \mid R \vee S : x < b[i])$

(8.19) **Axiom, Interchange of dummies:** Provided $\star$ is idempotent or R and Q are finite,
$\neg occurs(\text{`}y\text{'}, \text{`}R\text{'}), \neg occurs(\text{`}x\text{'}, \text{`}Q\text{'})$,
$(\star x \mid R : (\star y \mid Q : P)) = (\star y \mid Q : (\star x \mid R : P))$

$R:\ 1 \le x < 4$
$Q:\ 8 \le y < 10$
$P:\ 6 \cdot x \cdot y$

Note that $\neg occurs(\text{`}y\text{'}, \text{`}1 \le x < 4\text{'})$ and $\neg occurs(\text{`}x\text{'}, \text{`}8 \le y < 10\text{'})$

$(+x \mid R : (+y \mid Q : 6 \cdot x \cdot y))$
$=$ $\langle$Expand inner quantification$\rangle$
$(+x \mid R : 6 \cdot x \cdot 8 + 6 \cdot x \cdot 9)$
$=$ $\langle$Expand quantification$\rangle$
$(6 \cdot 1 \cdot 8 + 6 \cdot 1 \cdot 9) + (6 \cdot 2 \cdot 8 + 6 \cdot 2 \cdot 9) + (6 \cdot 3 \cdot 8 + 6 \cdot 3 \cdot 9)$
$=$ $\langle$Symmetry and associativity of $+\rangle$
$(6 \cdot 1 \cdot 8 + 6 \cdot 2 \cdot 8 + 6 \cdot 3 \cdot 8) + (6 \cdot 1 \cdot 9 + 6 \cdot 2 \cdot 9 + 6 \cdot 3 \cdot 9)$
$=$ $\langle$Quantify over $x\rangle$
$(+x \mid R : 6 \cdot x \cdot 8) + (+x \mid R : 6 \cdot x \cdot 9)$
$=$ $\langle$Quantify over $y\rangle$
$(+y \mid Q : (+x \mid R : 6 \cdot x \cdot y))$