Mathematical Induction

$$P.n: \ (\Sigma i \mid 1 \leq i \leq n : 2 \cdot i - 1) = n^2$$

$P.1: \ 1 = 1^2$ $\quad$ *true*

$P.2: \ 1 + 3 = 2^2$ $\quad$ *true*

$P.3: \ 1 + 3 + 5 = 3^2$ $\quad$ *true*

How can you prove $P.n$ is true for all $n \geq 1$, not just $1 \leq n \leq 3$?

A Logical Approach to Discrete Math

Idea: Suppose you could prove $P.n \Rightarrow P(n+1)$ in general.
Then you could use (3.77) Modus ponens $p \wedge (p \Rightarrow q) \Rightarrow q$ as follows:

First prove $P.1$.
Then,
$P.1 \wedge (P.1 \Rightarrow P.2) \Rightarrow P.2$
$P.2 \wedge (P.2 \Rightarrow P.3) \Rightarrow P.3$
$P.3 \wedge (P.3 \Rightarrow P.4) \Rightarrow P.4$
...
Conclusion: $P.n$ is true for all $n \geq 1$.

Proving $P.1$ is called the base case.
Proving $P.n \Rightarrow P(n+1)$ by deduction is called the induction case.
The antecedent $P.n$, which you assume, is called the inductive hypothesis.

A Logical Approach to Discrete Math

Prove $(\Sigma i \mid 1 \leq i \leq n : 2 \cdot i - 1) = n^2$

Proof

<u>Base case</u>

$$
\begin{array}{ll}
 & (\Sigma i \mid 1 \leq i \leq n : 2 \cdot i - 1) = n^2 \\
= & \langle \text{Base case}, n = 1 \rangle \\
 & (\Sigma i \mid 1 \leq i \leq 1 : 2 \cdot i - 1) = 1^2 \\
= & \langle \text{Math} \rangle \\
 & 2 \cdot 1 - 1 = 1 \\
= & \langle \text{Math} \rangle \\
 & 1 = 1 \quad //
\end{array}
$$

Induction case

Prove $(\Sigma i \mid 1 \leq i \leq n+1 : 2 \cdot i - 1) = (n+1)^2$

assuming $(\Sigma i \mid 1 \leq i \leq n : 2 \cdot i - 1) = n^2$

as the inductive hypothesis.

$$
\begin{array}{ll}
 & (\Sigma i \mid 1 \leq i \leq n+1 : 2 \cdot i - 1) \\
= & \quad \langle \text{Split off last term} \rangle \\
 & (\Sigma i \mid 1 \leq i \leq n : 2 \cdot i - 1) + 2(n+1) - 1 \\
= & \quad \langle \text{Inductive hypothesis} \rangle \\
 & n^2 + 2(n+1) - 1 \\
= & \quad \langle \text{Math} \rangle \\
 & n^2 + 2n + 1 \\
= & \quad \langle \text{Math} \rangle \\
 & (n+1)^2 \quad //
\end{array}
$$

Prove $(\Sigma i \mid 0 \leq i < n : 2^i) = 2^n - 1 \quad \text{for } n \geq 0$

Proof

Base case

$$
\begin{array}{ll}
 & (\Sigma i \mid 0 \leq i < n : 2^i) = 2^n - 1 \\
= & \langle \text{Base case}, n = 0 \rangle \\
 & (\Sigma i \mid 0 \leq i < 0 : 2^i) = 2^0 - 1 \\
= & \langle \text{Math} \rangle \\
 & (\Sigma i \mid false : 2^i) = 2^0 - 1 \\
= & \langle \text{(8.13) Empty range rule, and math} \rangle \\
 & 0 = 0 \quad //
\end{array}
$$

A Logical Approach to Discrete Math

Prove $(\Sigma i \mid 0 \le i < n : 2^i) = 2^n - 1 \quad \text{for } n \ge 0$

<u>Induction case</u>

Prove $(\Sigma i \mid 0 \le i < n+1 : 2^i) = 2^{n+1} - 1$

assuming $(\Sigma i \mid 0 \le i < n : 2^i) = 2^n - 1$

as the inductive hypothesis.

$$
\begin{array}{ll}
 & (\Sigma i \mid 0 \le i < n+1 : 2^i) \\
= & \langle \text{Split off last term} \rangle \\
 & (\Sigma i \mid 0 \le i < n : 2^i) + 2^n \\
= & \langle \text{Inductive hypothesis} \rangle \\
 & 2^n - 1 + 2^n \\
= & \langle \text{Math} \rangle \\
 & 2 \cdot 2^n - 1 \\
= & \langle \text{Math} \rangle \\
 & 2^{n+1} - 1 \quad //
\end{array}
$$

Prove $(\Sigma i \mid 0 \le i < n : 3^i) = (3^n - 1)/2$ for $n \ge 0$

Proof

Base case

$$
\begin{aligned}
& (\Sigma i \mid 0 \le i < n : 3^i) = (3^n - 1)/2 \\
= \quad & \langle \text{Base case}, n = 0 \rangle \\
& (\Sigma i \mid 0 \le i < 0 : 3^i) = (3^0 - 1)/2 \\
= \quad & \langle \text{Math} \rangle \\
& (\Sigma i \mid false : 3^i) = (3^0 - 1)/2 \\
= \quad & \langle \text{(8.13) Empty range rule, and math} \rangle \\
& 0 = 0 \quad //
\end{aligned}
$$

Prove $(\Sigma i \mid 0 \leq i < n : 3^i) = (3^n - 1)/2 \quad$ for $n \geq 0$

Induction case

Prove $(\Sigma i \mid 0 \leq i < n+1 : 3^i) = (3^{n+1} - 1)/2$

assuming $(\Sigma i \mid 0 \leq i < n : 3^i) = (3^n - 1)/2$

as the inductive hypothesis.

$$
\begin{array}{ll}
 & (\Sigma i \mid 0 \leq i < n+1 : 3^i) \\
= & \langle \text{Split off last term} \rangle \\
 & (\Sigma i \mid 0 \leq i < n : 3^i) + 3^n \\
= & \langle \text{Inductive hypothesis} \rangle \\
 & (3^n - 1)/2 + 3^n \\
= & \langle \text{Math, common denominator} \rangle \\
 & (3^n - 1 + 2 \cdot 3^n)/2 \\
= & \langle \text{Math} \rangle \\
 & (3 \cdot 3^n - 1)/2 \\
= & \langle \text{Math} \rangle \\
 & (3^{n+1} - 1)/2 \quad //
\end{array}
$$

Prove $2n+1 < 2^n \quad \text{for } n \geq 3$

Proof

<u>Base case</u>

$$
\begin{array}{ll}
 & 2n+1 < 2^n \\
= & \langle \text{Base case}, n = 3 \rangle \\
 & 2 \cdot 3 + 1 < 2^3 \\
= & \langle \text{Math} \rangle \\
 & 7 < 8 \quad //
\end{array}
$$

Induction case

Prove $2(n+1)+1 < 2^{n+1}$

assuming $2n+1 < 2^n$

as the inductive hypothesis.

$$
\begin{array}{ll}
 & 2^{n+1} \\
= & \langle \text{Math} \rangle \\
 & 2 \cdot 2^n \\
> & \langle \text{Inductive hypothesis} \rangle \\
 & 2 \cdot (2n+1) \\
= & \langle \text{Math} \rangle \\
 & 2(n+1)+1+2n-1 \\
> & \langle 2n-1 \text{ is positive for } n \geq 3 \rangle \\
 & 2(n+1)+1 \quad //
\end{array}
$$

Example of a proof by induction. Consider a currency consisting of 2-cent and 5-cent coins. Show that any amount above 3 cents can be represented using these coins.

We write $P.n$ in English as

$$P.n : \text{Some bag of 2-cent and 5-cent coins has the sum } n\ .$$

Our task is to prove $(\forall n \mid 4 \leq n : P.n)$.

Prove any amount more than 3 cents is possible using only 2-cent and 5-cent coins.

<u>Base case</u>

The base case is n=4.

Must prove that you can make 4 cents using only 2-cent and 5-cent coins.

Use two 2-cent coins. //

Induction case

Must prove

"n+1 cents is possible with 2-cent and 5-cent coins"

assuming

"n cents is possible with 2-cent and 5-cent coins"

as the inductive hypothesis.

Case 1

You have n cents with at least one 5-cent coin. Remove one 5-cent coin and replace it with three 2-cent coins. Now you have n+1 cents with only 2-cent and 5-cent coins.

Case 2

You have no five cent coins. If you have no 5-cents coins, they must all be 2-cent coins. Because the amount must be more than three cents, you must have at least two 2-cent coins. Remove two 2-cent coins and replace them with one 5-cent coin. Now you have n+1 cents with only 2-cent and 5-cent coins. //

(12.11) **Definition, b to the power n:**

$b^0 = 1$

$b^{n+1} = b \cdot b^n \quad$ for $n \geq 0$

(12.12) b **to the power** n**:**

$b^0 = 1$

$b^n = b \cdot b^{n-1} \quad$ for $n \geq 1$

(12.13) **Definition, factorial:**

$0! = 1$

$n! = n \cdot (n-1)! \quad$ for $n > 0$

Prove $n! = (\Pi i \mid 1 \leq i \leq n : i) \quad \text{for } n \geq 0$

Proof

<u>Base case</u>

$$
\begin{aligned}
& n! = (\Pi i \mid 1 \leq i \leq n : i) \\
= \quad & \langle \text{Base case}, n = 0 \rangle \\
& 0! = (\Pi i \mid 1 \leq i \leq 0 : i) \\
= \quad & \langle (12.13) \text{ and math} \rangle \\
& 1 = (\Pi i \mid false : i) \\
= \quad & \langle (8.13) \text{ Empty range rule} \rangle \\
& 1 = 1 \quad //
\end{aligned}
$$

Prove $n! = (\Pi i \mid 1 \leq i \leq n : i) \quad$ for $n \geq 0$

<u>Induction case</u>

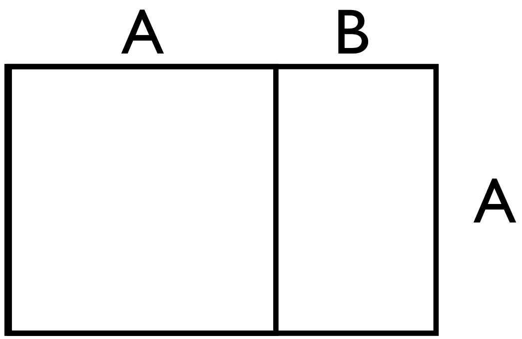
Prove $(n+1)! = (\Pi i \mid 1 \leq i \leq n+1 : i)$

assuming $n! = (\Pi i \mid 1 \leq i \leq n : i)$

as the inductive hypothesis.

$$
\begin{array}{ll}
& (\Pi i \mid 1 \leq i \leq n+1 : i) \\
= & \langle \text{Split off last term} \rangle \\
& (\Pi i \mid 1 \leq i \leq n : i) \cdot (n+1) \\
= & \langle \text{Inductive hypothesis} \rangle \\
& n! \cdot (n+1) \\
= & \langle (12.13 \text{ with } n := n+1, \text{ which is } (n+1)! = (n+1) \cdot n! \rangle \\
& (n+1)! \quad //
\end{array}
$$

The Golden Ratio



The golden ratio is $\phi = A/B$

By definition

$$\frac{A}{B} = \frac{A+B}{A}$$

$$\frac{A}{B} = 1 + \frac{B}{A}$$

$$\frac{A}{B} = 1 + \frac{1}{A/B}$$

$$\phi = 1 + \frac{1}{\phi}$$

$$\phi = 1 + \frac{1}{\phi}$$

$$\phi^2 = \phi + 1$$

$$\phi^2 - \phi - 1 = 0$$

$$\phi = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)}$$

$$\phi = \frac{1 \pm \sqrt{5}}{2}$$

$$\phi = \frac{1 + \sqrt{5}}{2} \qquad \hat{\phi} = \frac{1 - \sqrt{5}}{2}$$

The Fibonacci sequence

0	1	1	2	3	5	8
F_0	F_1	F_2	F_3	F_4	F_5	F_6

A Logical Approach to Discrete Math

(12.14) **Definition, Fibonacci:**

$F_0 = 0, \quad F_1 = 1$

$F_n = F_{n-1} + F_{n-2} \quad$ for $n > 1$

(12.14.1) **Definition, Golden Ratio:** $\phi = (1 + \sqrt{5})/2 \approx 1.618 \qquad \hat{\phi} = (1 - \sqrt{5})/2 \approx -0.618$

(12.15) $\phi^2 = \phi + 1$ and $\hat{\phi}^2 = \hat{\phi} + 1$

(12.16) $F_n \leq \phi^{n-1}$ for $n \geq 1$

(12.16.1) $\phi^{n-2} \leq F_n$ for $n \geq 1$

(12.17) $F_{n+m} = F_m \cdot F_{n+1} + F_{m-1} \cdot F_n$ for $n \geq 0$ and $m \geq 1$

To prove Fibonacci theorems
there are two base cases
and two inductive hypotheses.

A Logical Approach to Discrete Math

Prove (12.16) $F_n \leq \phi^{n-1}$ for $n \geq 1$

Proof

First base case

$$
\begin{array}{ll}
& F_n \leq \phi^{n-1} \\
= & \langle \text{Base case}, n = 1 \rangle \\
& F_1 \leq \phi^{1-1} \\
= & \langle (12.14) \rangle \\
& 1 \leq \phi^{1-1} \\
= & \langle \text{Math} \rangle \\
& 1 \leq 1 \quad //
\end{array}
$$

Second base case

$$
\begin{array}{ll}
& F_n \leq \phi^{n-1} \\
= & \langle \text{Base case}, n = 2 \rangle \\
& F_2 \leq \phi^{2-1} \\
= & \langle (12.14) \text{ and math} \rangle \\
& 1 + 0 \leq \phi \\
= & \langle (12.14.1) \text{ and math} \rangle \\
& 1 \leq 1.618 \quad //
\end{array}
$$

Prove (12.16) $\quad F_n \leq \phi^{n-1} \quad$ for $n \geq 1$

Induction case

Prove $F_{n+1} \leq \phi^{(n+1)-1}$

assuming $F_n \leq \phi^{n-1}$ and $F_{n-1} \leq \phi^{(n-1)-1}$

as the inductive hypotheses.

$$\begin{array}{ll} & F_{n+1} \\ = & \quad \langle (12.14) \text{ with } n := n+1 \rangle \\ & F_n + F_{n-1} \\ \leq & \quad \langle \text{Inductive hypotheses} \rangle \\ & \phi^{n-1} + \phi^{n-2} \\ = & \quad \langle \text{Math, factor out } \phi^{n-2} \rangle \\ & \phi^{n-2}(\phi + 1) \\ = & \quad \langle (12.15) \rangle \\ & \phi^{n-2} \cdot \phi^2 \\ = & \quad \langle \text{Math} \rangle \\ & \phi^{(n+1)-1} \quad // \end{array}$$

Inductively defined binary trees.

(12.30) **Definition, Binary Tree:**

$\emptyset$ is a binary tree, called the empty tree.

(d, l, r) is a binary tree, for d: $\mathbb{Z}$ and l, r binary trees.

$$\varnothing$$

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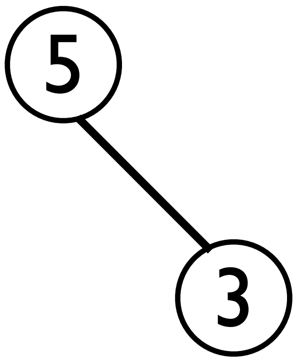

$(5, \emptyset, (3, \emptyset, \emptyset))$

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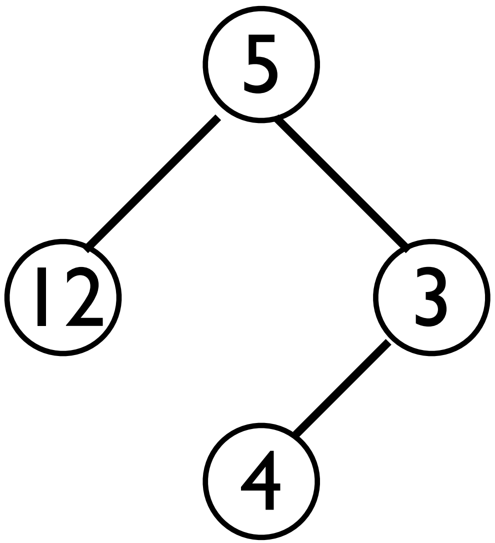

(5, (12, ∅, ∅), (3, (4, ∅, ∅), ∅))

(12.31) **Definition, Number of Nodes:**

$\#\emptyset = 0$

$\#(d,l,r) = 1 + \#l + \#r$

(12.32) **Definition, Height:**

$height.\emptyset = 0$

$height.(d,l,r) = 1 + max(height.l, height.r)$

(12.32.1) **Definition, Leaf:** A leaf is a node with no children (i.e. two empty subtrees).

(12.32.2) **Definition, Internal node:** An internal node is a node that is not a leaf.

(12.32.3) **Definition, Complete:** A binary tree is complete if every node has either 0 or 2 children.

$(5, \emptyset, \emptyset)$

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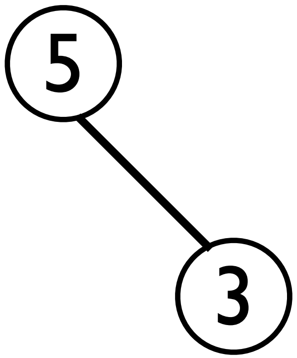

(5, ∅, (3, ∅, ∅))

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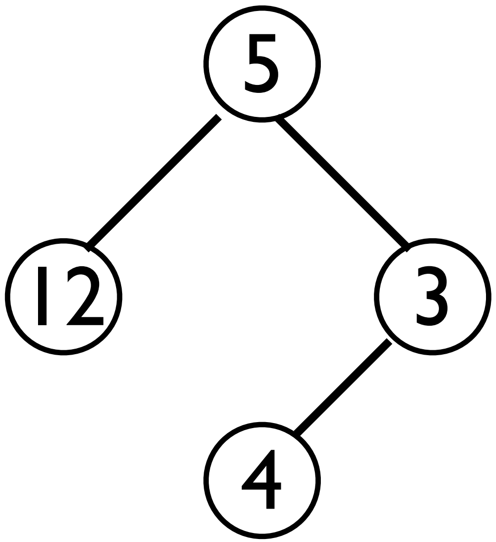

(5, (12, ∅, ∅), (3, (4, ∅, ∅), ∅))

(12.33) The maximum number of nodes in a tree with height n is $2^n - 1$ for $n \geq 0$.

(12.34) The minimum number of nodes in a tree with height n is n for $n \geq 0$.

(12.35) (a) The maximum number of leaves in a tree with height n is 2^{n-1} for $n > 0$.

(b) The maximum number of internal nodes is $2^{n-1} - 1$ for $n > 0$.

(12.36) (a) The minimum number of leaves in a tree with height n is 1 for $n > 0$.

(b) The minimum number of internal nodes is $n - 1$ for $n > 0$.

(12.37) Every nonempty complete tree has an odd number of nodes.

Prove (12.33) the maximum number of nodes in a tree of height n is $2^n - 1$.

<u>Base case</u>

(a) The empty tree has zero nodes.

(b) $2^0 - 1 = 0$ //

Prove (12.33) the maximum number of nodes in a tree of height n is $2^n - 1$.

<u>Induction case</u>

Must prove "the maximum number of nodes in a tree of height n+1 is $2^{n+1} - 1$"
assuming
"the maximum number of nodes in a tree of height n is $2^n - 1$" as the inductive hypothesis.

Proof: A tree height n+1 with the maximum number of nodes must have two children of height n, each with the maximum number of nodes.

By the inductive hypothesis
l has 2^n-1 nodes
and r has 2^n-1 nodes.
So, including the root node,
the total is

$1 + (\# \text{ in } l) + (\# \text{ in } r)$

= <Ind. hyp.>

$1 + 2^n-1 + 2^n-1$

= <Math>

$1 + 2 \times 2^n - 2$

= <Math>

$2^{n+1} - 1$ //