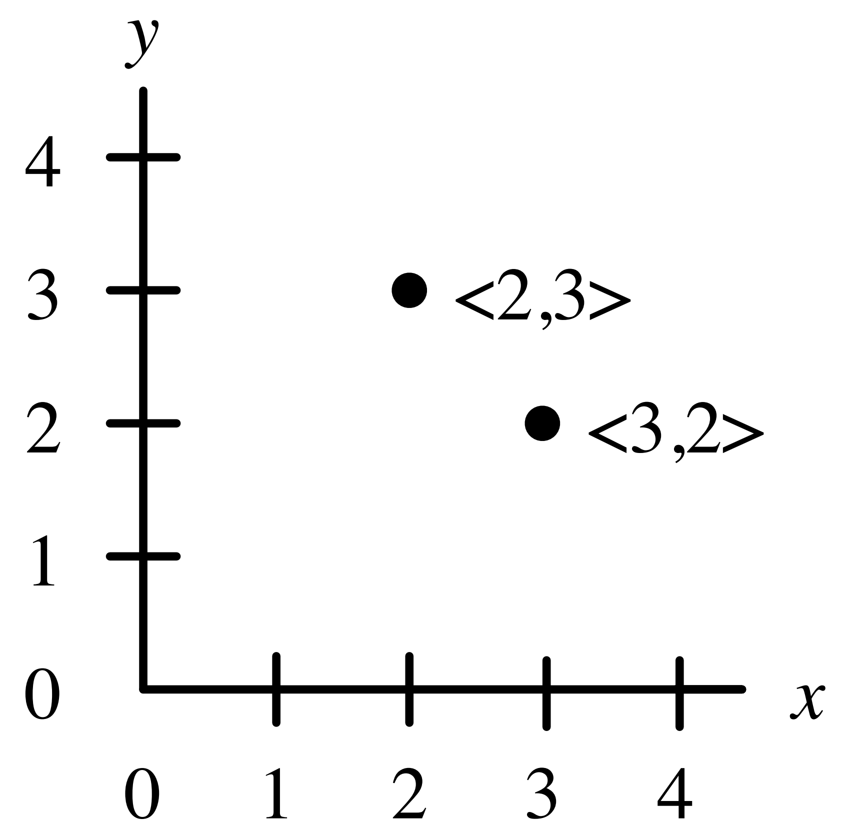
# A Logical Approach to Discrete Math

(14.2) **Axiom, Pair equality:** $\langle b,c\rangle = \langle b',c'\rangle \equiv b = b' \wedge c = c'$

(14.2.1) **Ordered pair one-point rule:** Provided $\neg occurs(‘x,y’, ‘E,F’)$,
$(\star x,y \mid \langle x,y\rangle = \langle E,F\rangle : P) = P[x,y := E,F]$ Homework

Sets: $\{2,3\} = \{3,2\}$

Ordered pairs: $\langle 2,3\rangle \neq \langle 3,2\rangle$



(14.3) **Axiom, Cross product:** $S \times T = \{b, c \mid b \in S \land c \in T : \langle b, c \rangle\}$

<u>Example</u>

$S = \{a, b, c\}$

$T = \{4, 6\}$

$S \times T = \{\langle a, 4\rangle, \langle a, 6\rangle, \langle b, 4\rangle, \langle b, 6\rangle, \langle c, 4\rangle, \langle c, 6\rangle\}$

$\mathbb{R} \times \mathbb{R}$ is the set of all points in the plane.

(11.4) **Axiom, Extensionality:** $S = T \equiv (\forall x \mid : x \in S \equiv x \in T)$

(14.3.1) **Axiom, Ordered pair extensionality:**
$U = V \equiv (\forall x, y \mid : \langle x, y \rangle \in U \equiv \langle x, y \rangle \in V)$

$U$ and $V$ are sets of ordered pairs.

Example

These two sets are equal.

$U = \{\langle 1, 3 \rangle, \langle 5, 0 \rangle, \langle 4, 2 \rangle\}$

$V = \{\langle 4, 2 \rangle, \langle 1, 3 \rangle, \langle 5, 0 \rangle\}$

## Relations and Functions

(14.2) **Axiom, Pair equality:** $\langle b, c\rangle = \langle b', c'\rangle \;\equiv\; b = b' \wedge c = c'$

(14.2.1) **Ordered pair one-point rule:** Provided $\neg occurs($'$x, y$', '$E, F$'$)$,
$(\star x, y \mid \langle x, y\rangle = \langle E, F\rangle : P) = P[x, y := E, F]$

(14.3) **Axiom, Cross product:** $S \times T = \{b, c \mid b \in S \wedge c \in T : \langle b, c\rangle\}$

(14.3.1) **Axiom, Ordered pair extensionality:**
$U = V \;\equiv\; (\forall x, y \mid : \langle x, y\rangle \in U \equiv \langle x, y\rangle \in V)$

**Theorems for cross product.**

(14.4) **Membership:** $\langle x, y\rangle \in S \times T \;\equiv\; x \in S \wedge y \in T$ Homework

(14.5) $\langle x, y\rangle \in S \times T \;\equiv\; \langle y, x\rangle \in T \times S$ Homework

(14.6) $S = \emptyset \Rightarrow S \times T = T \times S = \emptyset$

(14.7) $S \times T = T \times S \;\equiv\; S = \emptyset \vee T = \emptyset \vee S = T$

(14.8) **Distributivity of $\times$ over $\cup$ :**

(a) $S \times (T \cup U) = (S \times T) \cup (S \times U)$

(b) $(S \cup T) \times U = (S \times U) \cup (T \times U)$

(14.9) **Distributivity of $\times$ over $\cap$ :**

(a) $S \times (T \cap U) = (S \times T) \cap (S \times U)$

(b) $(S \cap T) \times U = (S \times U) \cap (T \times U)$

(14.10) **Distributivity of $\times$ over $-$ :**

$S \times (T - U) = (S \times T) - (S \times U)$

(14.11) **Monotonicity:** $T \subseteq U \ \Rightarrow\ S \times T \subseteq S \times U$

(14.12) $S \subseteq U \wedge T \subseteq V \ \Rightarrow\ S \times T \subseteq U \times V$

(14.13) $S \times T \subseteq S \times U \wedge S \neq \emptyset \ \Rightarrow\ T \subseteq U$

(14.14) $(S \cap T) \times (U \cap V) = (S \times U) \cap (T \times V)$

(14.15) For finite $S$ and $T$, $\#(S \times T) = \#S \cdot \#T$

# A Logical Approach to Discrete Math

Prove (14.8a) $S \times (T \cup U) = (S \times T) \cup (S \times U)$

*Proof*

Let $\langle x, y\rangle$ be an arbitrary ordered pair and prove that

$\langle x, y\rangle \in S \times (T \cup U) \;\equiv\; \langle x, y\rangle \in (S \times T) \cup (S \times U)$

$$
\begin{aligned}
& \langle x, y\rangle \in S \times (T \cup U) \\
= \quad & \langle (14.4) \rangle \\
& x \in S \wedge y \in (T \cup U) \\
= \quad & \langle (11.20) \rangle \\
& x \in S \wedge (y \in T \vee y \in U) \\
= \quad & \langle (3.46) \text{ Distributivity of } \wedge \text{ over } \vee \rangle \\
& (x \in S \wedge y \in T) \vee (x \in S \wedge y \in U) \\
= \quad & \langle (14.4 \text{ twice}) \rangle \\
& \langle x, y\rangle \in (S \times T) \vee \langle x, y\rangle \in (S \times U) \\
= \quad & \langle (11.20) \rangle \\
& \langle x, y\rangle \in (S \times T) \cup (S \times U) \qquad //
\end{aligned}
$$

**Relations.**

(14.15.1) **Definition, Binary relation:**

A *binary relation* over $B \times C$ is a subset of $B \times C$.

<u>Example</u>

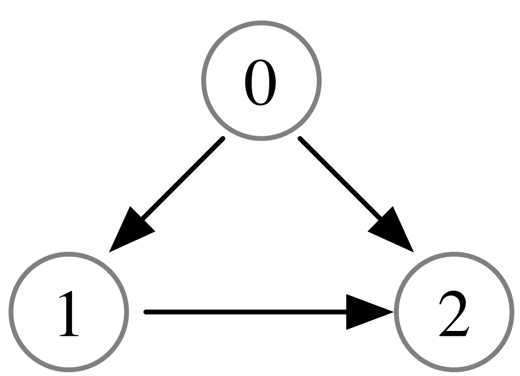
$S = \{0, 1, 2\}$

$$\begin{aligned} S \times S = \{ & \langle 0,0\rangle, \langle 0,1\rangle, \langle 0,2\rangle, \\ & \langle 1,0\rangle, \langle 1,1\rangle, \langle 1,2\rangle, \\ & \langle 2,0\rangle, \langle 2,1\rangle, \langle 2,2\rangle \} \end{aligned}$$

The "less than" relation over $S \times S$ is a subset of the set $S \times S$ consisting of those ordered pairs $\langle x, y\rangle$ for which $x < y$ is true.

$< \; = \; \{\langle 0,1\rangle, \langle 0,2\rangle, \langle 1,2\rangle\}$

<u>Directed graph representation</u>




<u>Matrix representation</u>

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \leftarrow \langle 1,2\rangle$$

(14.15.2) **Definition, Identity:** The identity relation $i_B$ on $B$ is $i_B = \{x\colon B \mid\colon \langle x, x \rangle\}$

(14.15.3) **Identity lemma:** $\langle x, y \rangle \in i_B \equiv x = y$ Homework

Example

$B = \{a, b, c, d\}$

The identity relation over $B \times B$ is

$i_B = \{\langle a, a \rangle, \langle b, b \rangle, \langle c, c \rangle, \langle d, d \rangle\}$

Matrix representation

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(14.15.4) **Notation:** $\langle b,c\rangle \in \rho$ and $b\ \rho\ c$ are interchangeable notations.

(14.15.5) **Conjunctive meaning:** $b\ \rho\ c\ \sigma\ d \ \equiv\ b\ \rho\ c \wedge c\ \sigma\ d$

<u>(14.15.4) Example</u>

If $\rho$ is the less than relation $<$ then
$\langle 0,2\rangle \in <$ and $0<2$ are interchangeable notations.

<u>(14.15.5) Example</u>

If $\rho$ is the less than relation $<$ and $\sigma$ is the equals relation $=$ then
$b<c=d \ \equiv\ b<c \wedge c=d$

# A Logical Approach to Discrete Math

The *domain* $Dom.\rho$ and *range* $Ran.\rho$ of a relation $\rho$ on $B \times C$ are defined by

(14.16) **Definition, Domain:** $Dom.\rho = \{b : B \mid (\exists c \mid : b\ \rho\ c)\}$

(14.17) **Definition, Range:** $Ran.\rho = \{c : C \mid (\exists b \mid : b\ \rho\ c)\}$

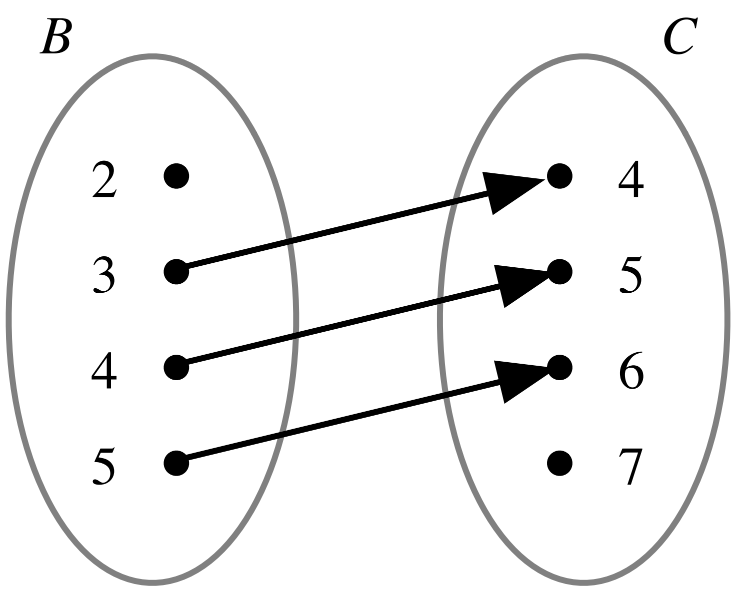
Example

$B = \{2,3,4,5\}$

$C = \{4,5,6,7\}$

Define the predecessor relation *pred* over $B \times C$ as

$pred = \{\langle 3,4\rangle, \langle 4,5\rangle, \langle 5,6\rangle\}$




$Dom.pred = \{3,4,5\}$

$Ran.pred = \{4,5,6\}$

The *inverse* $\rho^{-1}$ of a relation $\rho$ on $B \times C$ is the relation defined by

(14.18) **Definition, Inverse:** $\langle b, c\rangle \in \rho^{-1} \equiv \langle c, b\rangle \in \rho$, for all $b$: $B$, $c$: $C$

<u>Example</u>

$S = \{0, 1, 2\}$

The "less than" relation over $S \times S$ is

$$< \; = \; \{\langle 0,1\rangle, \langle 0,2\rangle, \langle 1,2\rangle\}$$

The inverse of the "less than" relation is

$$<^{-1} \; = \; \{\langle 1,0\rangle, \langle 2,0\rangle, \langle 2,1\rangle\}$$

which is the "greater than" relation $>$.

$$<^{-1} \; = \; >$$

## Operations on relations

Because $\rho$ and $\sigma$ are sets, you can operate on them with $\cup$, $\cap$, $\sim$, $-$.

Example

$B = \{0,1,2\}$

$<$ is $\{\langle 0,1\rangle, \langle 0,2\rangle, \langle 1,2\rangle\}$

$=$ is $\{\langle 0,0\rangle, \langle 1,1\rangle, \langle 2,2\rangle\}$

$< \cup =$ is $\{\langle 0,0\rangle, \langle 1,1\rangle, \langle 2,2\rangle, \langle 0,1\rangle, \langle 0,2\rangle, \langle 1,2\rangle\}$ which is $\leq$.

$\sim<$ is $\{\langle 0,0\rangle, \langle 1,0\rangle, \langle 1,1\rangle, \langle 2,0\rangle, \langle 2,1\rangle, \langle 2,2\rangle\}$ which is $\geq$.

$\leq \cap =$ is $=$.

$\leq - =$ is $<$.

(14.19) Let $\rho$ and $\sigma$ be relations.

(a) $Dom(\rho^{-1}) = Ran.\rho$ Homework

(b) $Ran(\rho^{-1}) = Dom.\rho$

(c) If $\rho$ is a relation on $B \times C$, then $\rho^{-1}$ is a relation on $C \times B$

(d) $(\rho^{-1})^{-1} = \rho$ Homework

(e) $\rho \subseteq \sigma \;\equiv\; \rho^{-1} \subseteq \sigma^{-1}$ Homework

Let $\rho$ be a relation on $B \times C$ and $\sigma$ be a relation on $C \times D$. The *product* of $\rho$ and $\sigma$, denoted by $\rho \circ \sigma$, is the relation on $B \times D$ defined by

(14.20) **Definition, Product:** $\langle b,d \rangle \in \rho \circ \sigma \equiv (\exists c \mid c \in C : \langle b,c \rangle \in \rho \wedge \langle c,d \rangle \in \sigma)$

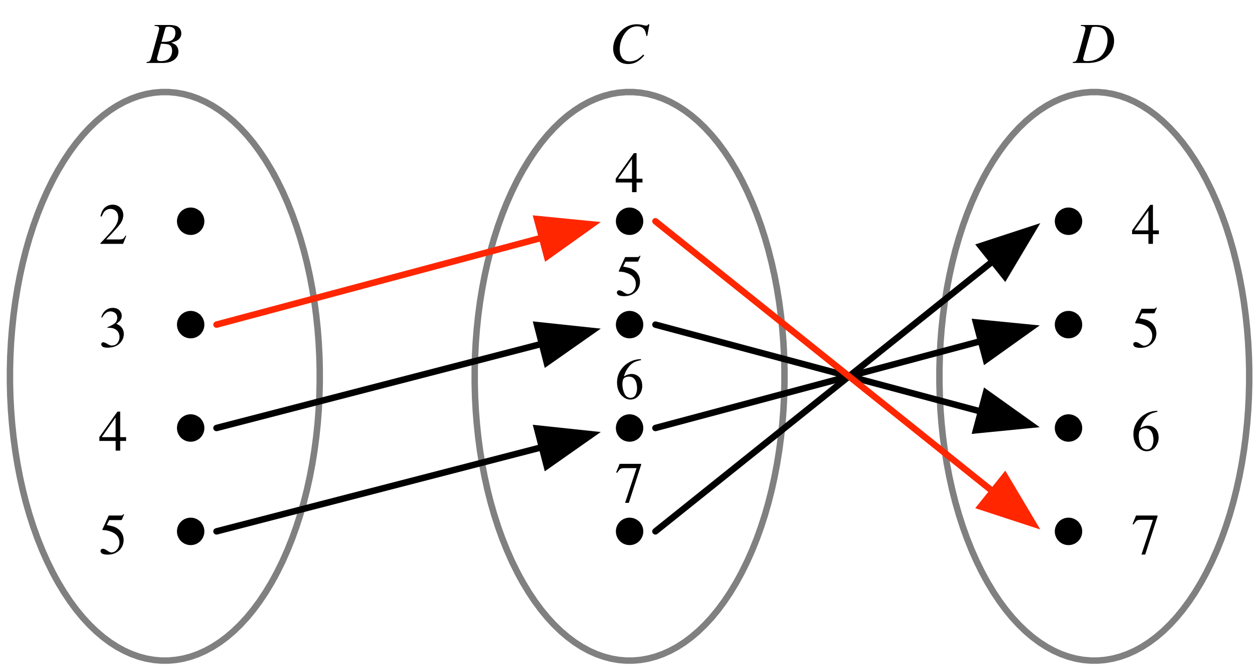
or, using the alternative notation by

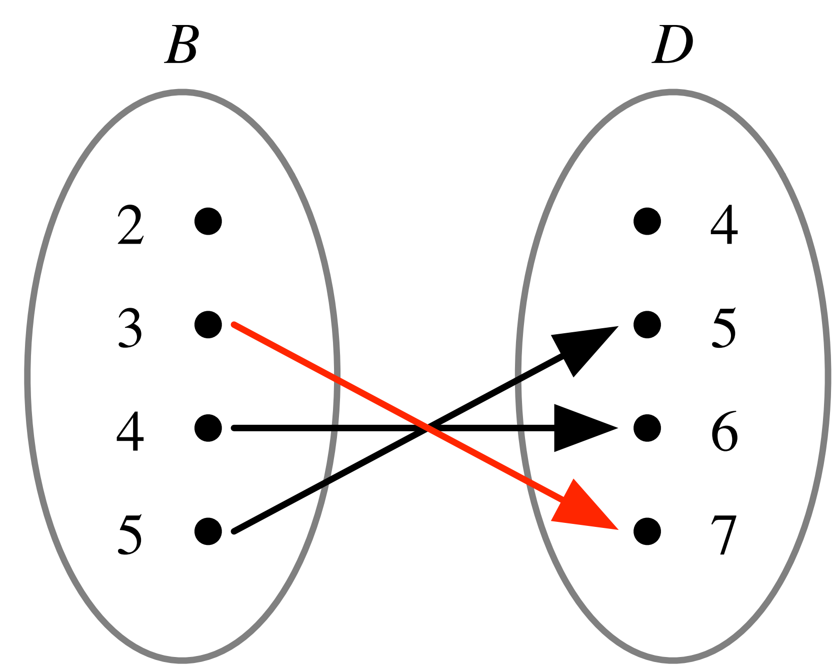
(14.21) **Definition, Product:** $b\ (\rho \circ \sigma)\ d \equiv (\exists c \mid : b\ \rho\ c\ \sigma\ d)$

$B = \{2,3,4,5\}$ $\quad pred = \{\langle 3,4 \rangle, \langle 4,5 \rangle, \langle 5,6 \rangle\}$ $\quad pred \circ swap = \{\langle 3,7 \rangle, \langle 4,6 \rangle, \langle 5,5 \rangle\}$

$C = \{4,5,6,7\}$ $\quad swap = \{\langle 4,7 \rangle, \langle 5,6 \rangle, \langle 6,5 \rangle, \langle 7,4 \rangle\}$

$D = \{4,5,6,7\}$




$Dom.pred = \{3,4,5\}$ $\quad Dom.swap = \{4,5,6,7\}$ $\quad Dom.(pred \circ swap) = \{3,4,5\}$

$Ran.pred = \{4,5,6\}$ $\quad Ran.swap = \{4,5,6,7\}$ $\quad Ran.(pred \circ swap) = \{5,6,7\}$

**Theorems for relation product.**

(14.22) **Associativity of $\circ$ :** $\rho \circ (\sigma \circ \theta) = (\rho \circ \sigma) \circ \theta$ Handout

(14.23) **Distributivity of $\circ$ over $\cup$ :**

(a) $\rho \circ (\sigma \cup \theta) = (\rho \circ \sigma) \cup (\rho \circ \theta)$ Homework

(b) $(\sigma \cup \theta) \circ \rho = (\sigma \circ \rho) \cup (\theta \circ \rho)$

(14.24) **Distributivity of $\circ$ over $\cap$ :**

(a) $\rho \circ (\sigma \cap \theta) \subseteq (\rho \circ \sigma) \cap (\rho \circ \theta)$

(b) $(\sigma \cap \theta) \circ \rho \subseteq (\sigma \circ \rho) \cap (\theta \circ \rho)$

(14.25) **Definition:**

$$\rho^0 = i_B$$

$$\rho^{n+1} = \rho^n \circ \rho \quad \text{for } n \geq 0$$

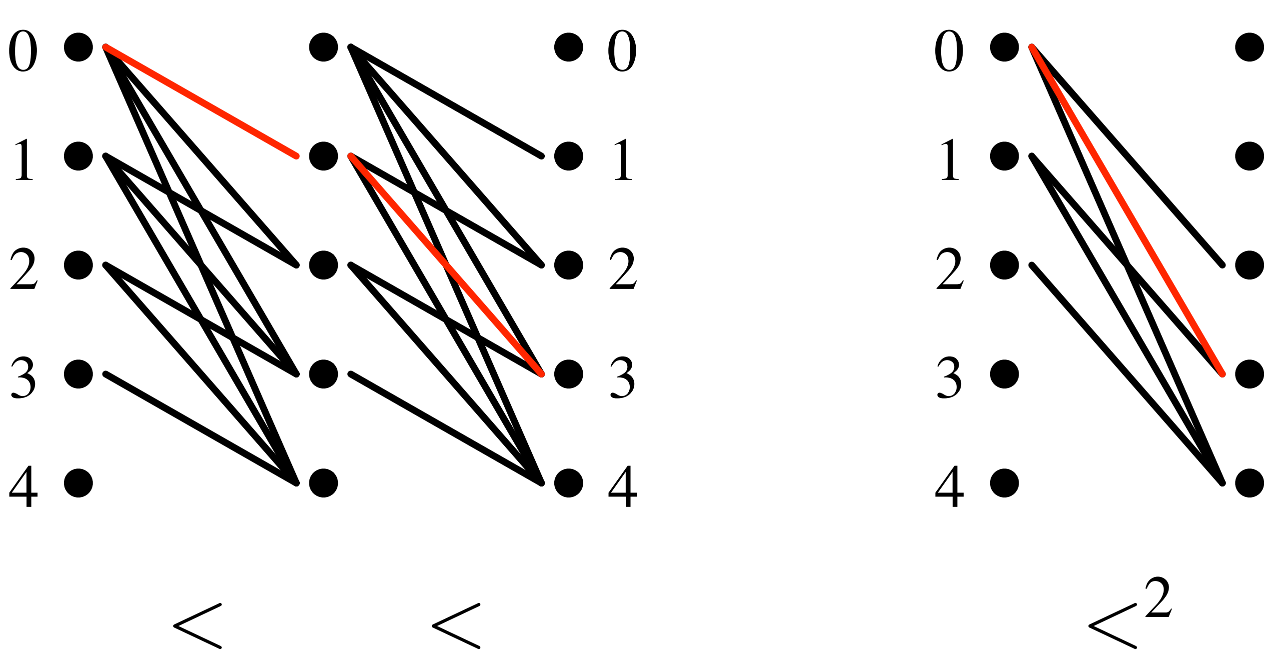
Example

$B = \{0, 1, 2, 3, 4\}$

$B \times B = \{\langle 0,0\rangle, \langle 0,1\rangle, ..., \langle 4,3\rangle, \langle 4,4\rangle\}$

$< \; = \; \{\langle 0,1\rangle, \langle 0,2\rangle, \langle 0,3\rangle, \langle 0,4\rangle, \langle 1,2\rangle, \langle 1,3\rangle, \langle 1,4\rangle, \langle 2,3\rangle, \langle 2,4\rangle, \langle 3,4\rangle\}$

$<^2 = < \circ < = \; \{\langle 0,2\rangle, \langle 0,3\rangle, \langle 0,4\rangle, \langle 1,3\rangle, \langle 1,4\rangle, \langle 2,4\rangle\}$



(14.25) **Definition:**

$$\rho^0 = i_B$$

$$\rho^{n+1} = \rho^n \circ \rho \quad \text{for } n \geq 0$$

Example

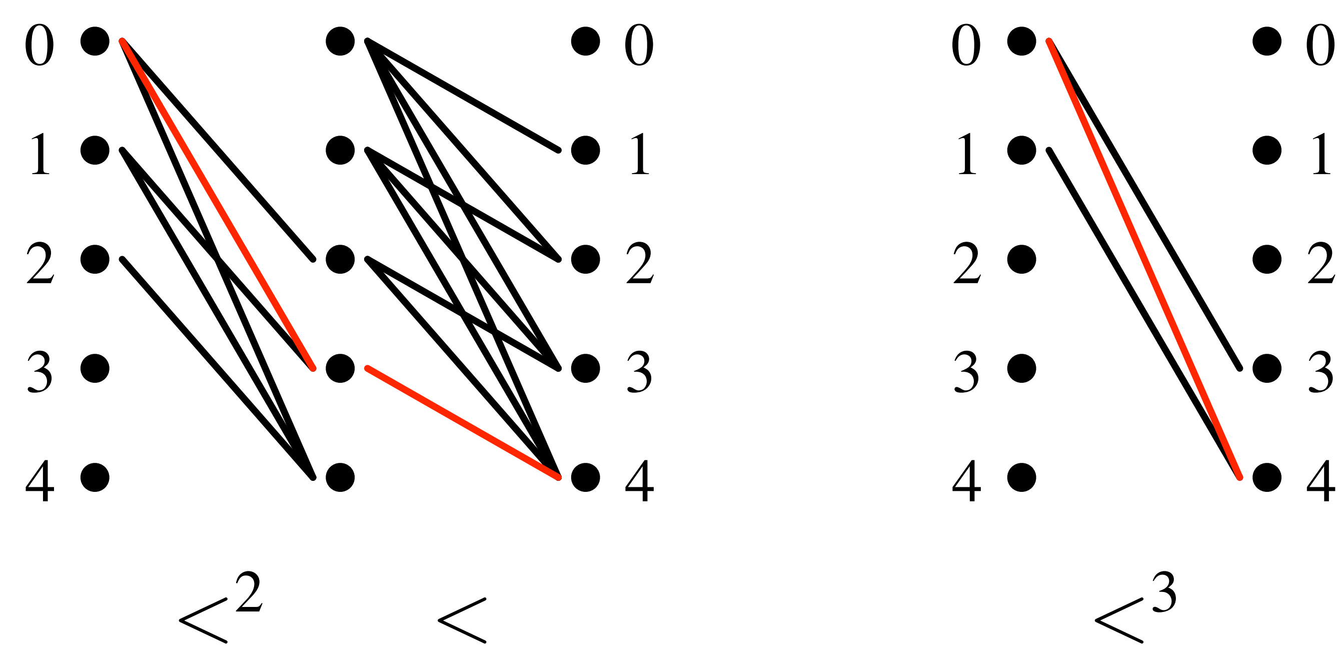
$B = \{0,1,2,3,4\}$

$B \times B = \{\langle 0,0\rangle, \langle 0,1\rangle, ..., \langle 4,3\rangle, \langle 4,4\rangle\}$

$< \; = \; \{\langle 0,1\rangle, \langle 0,2\rangle, \langle 0,3\rangle, \langle 0,4\rangle, \langle 1,2\rangle, \langle 1,3\rangle, \langle 1,4\rangle, \langle 2,3\rangle, \langle 2,4\rangle, \langle 3,4\rangle\}$

$<^2 \; = \; < \circ < \; = \; \{\langle 0,2\rangle, \langle 0,3\rangle, \langle 0,4\rangle, \langle 1,3\rangle, \langle 1,4\rangle, \langle 2,4\rangle\}$

$<^3 \; = \; <^2 \circ < \; = \; \{\langle 0,3\rangle, \langle 0,4\rangle, \langle 1,4\rangle\}$



(14.25) **Definition:**

$$\rho^0 = i_B$$

$$\rho^{n+1} = \rho^n \circ \rho \quad \text{for } n \geq 0$$

Example

$B = \{0, 1, 2\}$

$B \times B = \{\langle 0,0\rangle, \langle 0,1\rangle, \langle 0,2\rangle, \langle 1,0\rangle, \langle 1,1\rangle, \langle 1,2\rangle, \langle 2,0\rangle, \langle 2,1\rangle, \langle 2,2\rangle\}$

$\leq \; = \; \{\langle 0,0\rangle, \langle 0,1\rangle, \langle 0,2\rangle, \langle 1,1\rangle, \langle 1,2\rangle, \langle 2,2\rangle\}$

$\leq^2 = \leq \circ \leq = \; \{\langle 0,0\rangle, \langle 0,1\rangle, \langle 0,2\rangle, \langle 1,1\rangle, \langle 1,2\rangle, \langle 2,2\rangle\}$

$\leq \circ \leq = \leq$ Idempotent

**Table 14.1** Classes of relations $\rho$ over set $B$

| | Name | Property | Alternative |
|---|---|---|---|
| (a) | reflexive | $(\forall b \mid : b \; \rho \; b)$ | $i_B \subseteq \rho$ |
| (b) | irreflexive | $(\forall b \mid : \neg(b \; \rho \; b))$ | $i_B \cap \rho = \emptyset$ |
| (c) | symmetric | $(\forall b, c \mid : b \; \rho \; c \;\equiv\; c \; \rho \; b)$ | $\rho^{-1} = \rho$ |
| (d) | antisymmetric | $(\forall b, c \mid : b \; \rho \; c \wedge c \; \rho \; b \Rightarrow b = c)$ | $\rho \cap \rho^{-1} \subseteq i_B$ |
| (e) | asymmetric | $(\forall b, c \mid : b \; \rho \; c \Rightarrow \neg(c \; \rho \; b))$ | $\rho \cap \rho^{-1} = \emptyset$ |
| (f) | transitive | $(\forall b, c, d \mid : b \; \rho \; c \wedge c \; \rho \; d \Rightarrow b \; \rho \; d)$ | $\rho = (\cup i \mid i > 0 : \rho^i)$ |

Memorize

**Table 14.1** Classes of relations $\rho$ over set $B$

| | Name | Property | Alternative |
|---|---|---|---|
| (a) | reflexive | $(\forall b \mid : b \ \rho \ b)$ | $i_B \subseteq \rho$ |
| (b) | irreflexive | $(\forall b \mid : \neg(b \ \rho \ b))$ | $i_B \cap \rho = \emptyset$ |
| (c) | symmetric | $(\forall b, c \mid : b \ \rho \ c \ \equiv \ c \ \rho \ b)$ | $\rho^{-1} = \rho$ |
| (d) | antisymmetric | $(\forall b, c \mid : b \ \rho \ c \wedge c \ \rho \ b \Rightarrow b = c)$ | $\rho \cap \rho^{-1} \subseteq i_B$ |
| (e) | asymmetric | $(\forall b, c \mid : b \ \rho \ c \Rightarrow \neg(c \ \rho \ b))$ | $\rho \cap \rho^{-1} = \emptyset$ |
| (f) | transitive | $(\forall b, c, d \mid : b \ \rho \ c \wedge c \ \rho \ d \Rightarrow b \ \rho \ d)$ | $\rho = (\cup i \mid i > 0 : \rho^i)$ |

Example

The $>$ relation over $\mathbb{Z}$

| | | |
|---|---|---|
| (a) | $b > b$ | No, $>$ is not reflexive |
| (b) | $\neg(b > b)$ | Yes, $>$ is irreflexive |
| (c) | $b > c \ \equiv \ c > b$ | No, $>$ is not symmetric |
| (d) | $b > c \wedge c > b \ \Rightarrow \ b = c$ | Yes, $>$ is antisymmetric because the antecedent is always false |
| (e) | $b > c \ \Rightarrow \ \neg(c > b)$ | Yes, $>$ is asymmetric |
| (f) | $b > c \wedge c > d \ \Rightarrow \ b > d$ | Yes, $>$ is transitive |

**Table 14.1** Classes of relations $\rho$ over set $B$

| | Name | Property | Alternative |
|---|---|---|---|
| (a) | reflexive | $(\forall b \mid : b \; \rho \; b)$ | $i_B \subseteq \rho$ |
| (b) | irreflexive | $(\forall b \mid : \neg(b \; \rho \; b))$ | $i_B \cap \rho = \emptyset$ |
| (c) | symmetric | $(\forall b, c \mid : b \; \rho \; c \;\equiv\; c \; \rho \; b)$ | $\rho^{-1} = \rho$ |
| (d) | antisymmetric | $(\forall b, c \mid : b \; \rho \; c \wedge c \; \rho \; b \Rightarrow b = c)$ | $\rho \cap \rho^{-1} \subseteq i_B$ |
| (e) | asymmetric | $(\forall b, c \mid : b \; \rho \; c \Rightarrow \neg(c \; \rho \; b))$ | $\rho \cap \rho^{-1} = \emptyset$ |
| (f) | transitive | $(\forall b, c, d \mid : b \; \rho \; c \wedge c \; \rho \; d \Rightarrow b \; \rho \; d)$ | $\rho = (\cup i \mid i > 0 : \rho^i)$ |

Example

The *square* relation over $\mathbb{Z}$

$square = \{\langle 0,0\rangle, \langle 1,1\rangle, \langle 2,4\rangle, \langle 3,9\rangle, \ldots\}$

(a) $b \; square \; b$ — No, *square* is not reflexive. It does not have $\langle 2,2\rangle$.

(b) $\neg(b \; square \; b)$ — No, *square* is not irreflexive. It has $\langle 1,1\rangle$.

**Table 14.1** Classes of relations $\rho$ over set $B$

| | Name | Property | Alternative |
|---|---|---|---|
| (a) | reflexive | $(\forall b \mid : b \ \rho \ b)$ | $i_B \subseteq \rho$ |
| (b) | irreflexive | $(\forall b \mid : \neg(b \ \rho \ b))$ | $i_B \cap \rho = \emptyset$ |
| (c) | symmetric | $(\forall b, c \mid : b \ \rho \ c \ \equiv \ c \ \rho \ b)$ | $\rho^{-1} = \rho$ |
| (d) | antisymmetric | $(\forall b, c \mid : b \ \rho \ c \wedge c \ \rho \ b \Rightarrow b = c)$ | $\rho \cap \rho^{-1} \subseteq i_B$ |
| (e) | asymmetric | $(\forall b, c \mid : b \ \rho \ c \Rightarrow \neg(c \ \rho \ b))$ | $\rho \cap \rho^{-1} = \emptyset$ |
| (f) | transitive | $(\forall b, c, d \mid : b \ \rho \ c \wedge c \ \rho \ d \Rightarrow b \ \rho \ d)$ | $\rho = (\cup i \mid i > 0 : \rho^i)$ |

**Reflexive relations** – A reflexive relation $\rho$ is defined as $(\forall b \mid : b \ \rho \ b)$, or, alternatively as $i_B \subseteq \rho$. In terms of the matrix, the diagonal must contain all 1's. Each underline entry _ in the matrix of the reflexive relation on the right represents either a one or a zero.

$$\begin{bmatrix} 1 & \_ & \_ & \_ \\ \_ & 1 & \_ & \_ \\ \_ & \_ & 1 & \_ \\ \_ & \_ & \_ & 1 \end{bmatrix}$$

**Irreflexive relations** – An irreflexive relation $\rho$ is defined as $(\forall b \mid : \neg(b \ \rho \ b))$ or, alternatively, as $i_B \cap \rho = \emptyset$. In terms of the matrix, the diagonal must contain all 0's. It is possible for a relation to be neither reflexive nor irreflexive. The first example is one such relation.

$$\begin{bmatrix} 0 & \_ & \_ & \_ \\ \_ & 0 & \_ & \_ \\ \_ & \_ & 0 & \_ \\ \_ & \_ & \_ & 0 \end{bmatrix}$$

**Table 14.1** Classes of relations $\rho$ over set $B$

| | Name | Property | Alternative |
|---|---|---|---|
| (a) | reflexive | $(\forall b \mid : b\ \rho\ b)$ | $i_B \subseteq \rho$ |
| (b) | irreflexive | $(\forall b \mid : \neg(b\ \rho\ b))$ | $i_B \cap \rho = \emptyset$ |
| (c) | symmetric | $(\forall b, c \mid : b\ \rho\ c \ \equiv\ c\ \rho\ b)$ | $\rho^{-1} = \rho$ |
| (d) | antisymmetric | $(\forall b, c \mid : b\ \rho\ c \wedge c\ \rho\ b \Rightarrow b = c)$ | $\rho \cap \rho^{-1} \subseteq i_B$ |
| (e) | asymmetric | $(\forall b, c \mid : b\ \rho\ c \Rightarrow \neg(c\ \rho\ b))$ | $\rho \cap \rho^{-1} = \emptyset$ |
| (f) | transitive | $(\forall b, c, d \mid : b\ \rho\ c \wedge c\ \rho\ d \Rightarrow b\ \rho\ d)$ | $\rho = (\cup i \mid i > 0 : \rho^i)$ |

**Symmetric relations** – A symmetric relation $\rho$ is defined as $(\forall b,c \mid : b\ \rho\ c \ \equiv\ c\ \rho\ b)$ or, alternatively, as $\rho^{-1} = \rho$. In terms of the matrix, it must be symmetric about the diagonal. For example, in the matrix on the right the 1 in the first row, third column represents ordered pair $\langle w,y \rangle$, and the 1 in the third row, first column represents ordered pair $\langle y,w \rangle$. The 0 in the second row, third column represents the *absence* of $\langle x,y \rangle$, and the 0 in the third row, second column represents the *absence* of $\langle y,x \rangle$.

$$\begin{bmatrix} \_ & 1 & 1 & 1 \\ 1 & \_ & 0 & 0 \\ 1 & 0 & \_ & 0 \\ 1 & 0 & 0 & \_ \end{bmatrix}$$

**Antisymmetric relations** – An antisymmetric relation $\rho$ is defined as $(\forall b,c \mid : b\ \rho\ c \wedge c\ \rho\ b \Rightarrow b = c)$ or, alternatively, as $\rho \cap \rho^{-1} \subseteq i_B$. In terms of the matrix, the diagonal elements can be either 0 or 1. If $b\ \rho\ b$ is true, then both the antecedent and consequent are true, and so the implication is true. If $b\ \rho\ b$ is false, then the antecedent is false, and so the implication is true. For the off-diagonal elements, where $b \neq c$, you cannot have both $b\ \rho\ c$ and $c\ \rho\ b$. However, you can have neither.

$$\begin{bmatrix} \_ & 1 & 1 & 1 \\ 0 & \_ & 0 & 0 \\ 0 & 0 & \_ & 0 \\ 0 & 0 & 1 & \_ \end{bmatrix}$$

**Table 14.1** Classes of relations $\rho$ over set $B$

| | Name | Property | Alternative |
|---|---|---|---|
| (a) | reflexive | $(\forall b \mid : b \; \rho \; b)$ | $i_B \subseteq \rho$ |
| (b) | irreflexive | $(\forall b \mid : \neg(b \; \rho \; b))$ | $i_B \cap \rho = \emptyset$ |
| (c) | symmetric | $(\forall b, c \mid : b \; \rho \; c \;\equiv\; c \; \rho \; b)$ | $\rho^{-1} = \rho$ |
| (d) | antisymmetric | $(\forall b, c \mid : b \; \rho \; c \wedge c \; \rho \; b \Rightarrow b = c)$ | $\rho \cap \rho^{-1} \subseteq i_B$ |
| (e) | asymmetric | $(\forall b, c \mid : b \; \rho \; c \Rightarrow \neg(c \; \rho \; b))$ | $\rho \cap \rho^{-1} = \emptyset$ |
| (f) | transitive | $(\forall b, c, d \mid : b \; \rho \; c \wedge c \; \rho \; d \Rightarrow b \; \rho \; d)$ | $\rho = (\cup i \mid i > 0 : \rho^i)$ |

**Asymmetric relations** – An asymmetric relation $\rho$ is defined as $(\forall b, c \mid : b \; \rho \; c \Rightarrow \neg(c \; \rho \; b))$ or, alternatively, as $\rho \cap \rho^{-1} = \emptyset$. In terms of the matrix, the diagonal elements must be 0. If $b \; \rho \; b$ were true, then the antecedent would be true and the consequent would be false, and so the implication would be false. For the off-diagonal elements, where $b \neq c$, if you have $b \; \rho \; c$ you cannot have $c \; \rho \; b$. Like an antisymmetric relation, you can have neither. An asymmetric relation is an antisymmetric relation with the added restriction that the diagonal elements must be 0.

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Prove Table 14.1(a) $\quad (\forall b \mid : b \, \rho \, b) \equiv i_B \subseteq \rho$

*Proof*

$$
\begin{array}{ll}
& i_B \subseteq \rho \\
= & \langle \text{(11.13) Axiom, Subset} \rangle \\
& (\forall b, c \mid \langle b, c \rangle \in i_B : \langle b, c \rangle \in \rho) \\
= & \langle \text{(14.15.3) Identity lemma} \rangle \\
& (\forall b, c \mid b = c : \langle b, c \rangle \in \rho) \\
= & \langle \text{(8.20) Nesting, with } R := true \rangle \\
& (\forall b \mid : (\forall c \mid b = c : \langle b, c \rangle \in \rho)) \\
= & \langle \text{(8.14) One-point rule and textual substitution} \rangle \\
& (\forall b \mid \langle b, b \rangle \in \rho) \\
= & \langle \text{(14.15.4) Notation} \rangle \\
& (\forall b \mid : b \, \rho \, b) \quad //
\end{array}
$$

(14.30.1) **Definition:** Let $\rho$ be a relation on a set. The *reflexive closure* of $\rho$ is the relation $r(\rho)$ that satisfies:

(a) $r(\rho)$ is reflexive;

(b) $\rho \subseteq r(\rho)$;

(c) If any relation $\sigma$ is reflexive and $\rho \subseteq \sigma$, then $r(\rho) \subseteq \sigma$.

Example

$B = \{0, 1, 2\}$

$< = \{\langle 0,1\rangle, \langle 0,2\rangle, \langle 1,2\rangle\}$

By part (b), every ordered pair in $<$ must also be in $r(<)$.

$r(<) = \{\langle 0,1\rangle, \langle 0,2\rangle, \langle 1,2\rangle, \ldots\}$

By part (a), $r(<)$ must be reflexive.

$r(<) = \{\langle 0,1\rangle, \langle 0,2\rangle, \langle 1,2\rangle, \langle 0,0\rangle, \langle 1,1\rangle, \langle 2,2\rangle, \ldots\}$

By part (c), there can be no other ordered pairs in $r(<)$.

$r(<) = \{\langle 0,1\rangle, \langle 0,2\rangle, \langle 1,2\rangle, \langle 0,0\rangle, \langle 1,1\rangle, \langle 2,2\rangle\}$

The relation

$\sigma = \{\langle 0,1\rangle, \langle 0,2\rangle, \langle 1,2\rangle, \langle 0,0\rangle, \langle 1,1\rangle, \langle 2,2\rangle \langle 1,0\rangle\}$

also satisfies (a) and (b) because (a) $\sigma$ is reflexive, and (b) $< \subseteq \sigma$.

However, $\sigma$ cannot be the reflexive closure of $<$, because $r(<) \subseteq \sigma$.

To compute $r(\rho)$, add the fewest number of ordered pairs to $\rho$ that will make it reflexive.

(14.30.2) **Definition:** Let $\rho$ be a relation on a set. The *symmetric closure* of $\rho$ is the relation $s(\rho)$ that satisfies:

(a) $s(\rho)$ is symmetric;

(b) $\rho \subseteq s(\rho)$;

(c) If any relation $\sigma$ is symmetric and $\rho \subseteq \sigma$, then $s(\rho) \subseteq \sigma$.

Example

$B = \{0,1,2\}$

$< = \{\langle 0,1\rangle, \langle 0,2\rangle, \langle 1,2\rangle\}$

$s(<) = \{\langle 0,1\rangle, \langle 0,2\rangle, \langle 1,2\rangle, \langle 1,0\rangle, \langle 2,0\rangle, \langle 2,1\rangle\}$

(14.30.3) **Definition:** Let $\rho$ be a relation on a set. The *transitive closure* of $\rho$ is the relation $\rho^+$ that satisfies:

(a) $\rho^+$ is transitive;

(b) $\rho \subseteq \rho^+$;

(c) If any relation $\sigma$ is transitive and $\rho \subseteq \sigma$, then $\rho^+ \subseteq \sigma$.

(14.30.4) **Definition:** Let $\rho$ be a relation on a set. The *reflexive transitive closure* of $\rho$ is the relation $\rho^*$ that is both the reflexive and the transitive closure of $\rho$.

Example

$B = \{0, 1, 2, 3\}$

$pred \ = \{\langle 0,1\rangle, \langle 1,2\rangle, \langle 2,3\rangle\}$

$pred^+ = \{$
$\quad \langle 0,1\rangle, \langle 1,2\rangle, \langle 2,3\rangle,$
$\quad \langle 0,2\rangle, \langle 1,3\rangle,$
$\quad \langle 0,3\rangle\}$

$pred^+ = \ <$

$pred^* = \{$
$\quad \langle 0,1\rangle, \langle 1,2\rangle, \langle 2,3\rangle, \langle 0,2\rangle, \langle 1,3\rangle, \langle 0,3\rangle,$
$\quad \langle 0,0\rangle\ \langle 1,1\rangle\ \langle 2,2\rangle\ \langle 3,3\rangle\}$

$pred^* = \ \leq$

Exercise 14.32

| | $\rho \cup \sigma$ | $\rho \cap \sigma$ | $\rho - \sigma$ | $(B \times B) - \rho$ |
|---|---|---|---|---|
| Reflexive | Y | | N | |
| Irreflexive | | | Y | |
| Symmetric | | | | |
| Antisymmetric | | | | |
| Transitive | | | | |

Is reflexivity preserved under union?

If $\rho$ is reflexive and $\sigma$ is reflexive, is $\rho \cup \sigma$ reflexive?

If $\rho$ has $\langle a,a\rangle, \langle b,b\rangle, \ldots$, and $\sigma$ has $\langle a,a\rangle, \langle b,b\rangle, \ldots$, does $\rho \cup \sigma$ have $\langle a,a\rangle, \langle b,b\rangle, \ldots$?

Is reflexivity preserved under set difference?

If $\rho$ is reflexive and $\sigma$ is reflexive, is $\rho - \sigma$ reflexive?

If $\rho$ has $\langle a,a\rangle, \langle b,b\rangle, \ldots$, and $\sigma$ has $\langle a,a\rangle, \langle b,b\rangle, \ldots$, does $\rho - \sigma$ have $\langle a,a\rangle, \langle b,b\rangle, \ldots$?

Is irreflexivity preserved under set difference?

If $\rho$ is irreflexive and $\sigma$ is irreflexive, is $\rho - \sigma$ irreflexive?

If $\rho$ and $\sigma$ are both missing $\langle a,a\rangle, \langle b,b\rangle, \ldots$, is $\rho - \sigma$ missing $\langle a,a\rangle, \langle b,b\rangle, \ldots$?

**Equivalence relations.**

(14.33) **Definition:** A relation is an *equivalence relation* iff it is reflexive, symmetric, and transitive

(14.34) **Definition:** Let $\rho$ be an equivalence relation on $B$. Then $[b]_\rho$, the *equivalence class* of $b$, is the subset of elements of $B$ that are equivalent (under $\rho$) to $b$:
$x \in [b]_\rho \quad \equiv \quad x \, \rho \, b$

(14.33) Example

$B = \{0,1,2,3,4\}$
$\rho = \{$
$\langle 0,0\rangle, \langle 1,1\rangle, \langle 2,2\rangle, \langle 3,3\rangle, \langle 4,4\rangle,$
$\langle 0,1\rangle, \langle 1,0\rangle, \langle 0,3\rangle, \langle 3,0\rangle, \langle 0,4\rangle, \langle 4,0\rangle,$
$\langle 2,4\rangle, \langle 4,2\rangle\}$

(14.34) Example

$[0] = \{0,1,3\}$
$[1] = \{1,0,3\}$
$[2] = \{2,4\}$
$[3] = \{3,1,0\}$
$[4] = \{4,2\}$

Partition

$[0] \cap [2] = \emptyset$
$[0] \cup [2] = B$
$\{[0],[2]\}$ is a partition of $B$.
$\{\{0,1,3\},\{2,4\}\}$ is a partition of $B$.

(11.76) **Axiom, Partition:** Set $S$ partitions $T$ if

(i) the sets in $S$ are pairwise disjoint and

(ii) the union of the sets in $S$ is $T$, that is, if

$$(\forall u,v \mid u \in S \wedge v \in S \wedge u \neq v : u \cap v = \emptyset) \wedge (\cup u \mid u \in S : u) = T$$

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<u>Example</u>

$T: \{a,b,c,d,e,f\}$

$S: \{\{a,c\},\{b,e,f\},\{d\}\}$

$S$ partitions $T$.

(11.76) **Axiom, Partition:** Set $S$ partitions $T$ if

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(ii) the union of the sets in $S$ is $T$, that is, if

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Example

$T: \{a,b,c,d,e,f\}$

$S: \{\{a,c\},\{b,e,f\},\{d,e\}\}$

$S$ does not partition $T$ because $\{b,e,f\} \cap \{d,e\} \neq \emptyset$.

(11.76) **Axiom, Partition:** Set $S$ partitions $T$ if

(i) the sets in $S$ are pairwise disjoint and

(ii) the union of the sets in $S$ is $T$, that is, if

$$(\forall u,v \mid u \in S \land v \in S \land u \neq v : u \cap v = \emptyset) \land (\cup u \mid u \in S : u) = T$$

<u>Example</u>

$T:\ \{a,b,c,d,e,f\}$

$S:\ \{\{a,c\},\{e,f\},\{d\}\}$

$S$ does not partition $T$ because $\{a,c\} \cup \{e,f\} \cup \{d\} \neq T$.

(14.35) Let $\rho$ be an equivalence relation on $B$, and let $b, c$ be members of $B$. The following three predicates are equivalent:

(a) $b \rho c$

(b) $[b] \cap [c] \neq \emptyset$

(c) $[b] = [c]$

That is, $(b \rho c) \;=\; ([b] \cap [c] \neq \emptyset) \;=\; ([b] = [c])$

Example

Using the previous example, the following are all equivalent:

(a) $1 \rho 3$

(b) $[1] \cap [3] \neq \emptyset$

(c) $[1] = [3]$

because each one is $true$.

The following are all equivalent:

(a) $1 \rho 2$

(b) $[1] \cap [2] \neq \emptyset$

(c) $[1] = [2]$

because each one is $false$.

(14.35) Let $\rho$ be an equivalence relation on $B$, and let $b, c$ be members of $B$. The following three predicates are equivalent:

(a) $b \rho c$

(b) $[b] \cap [c] \neq \emptyset$

(c) $[b] = [c]$

That is, $(b \rho c) \;=\; ([b] \cap [c] \neq \emptyset) \;=\; ([b] = [c])$

Prove (14.35)

To prove (14.35), first prove each of the following three sub-theorems:

(a) $\Rightarrow$ (b)

(b) $\Rightarrow$ (c)

(c) $\Rightarrow$ (a)

Then by (3.82a) Transitivity, ((b) $\Rightarrow$ (c)) $\wedge$ ((c) $\Rightarrow$ (a)) $\Rightarrow$ ((b) $\Rightarrow$ (a))

Then by (3.80) Mutual implication, ((a) $\Rightarrow$ (b)) $\wedge$ ((b) $\Rightarrow$ (a)) $\equiv$ ((a) $\equiv$ (b))

And similarly for (a) $\equiv$ (c) and for (b) $\equiv$ (c)

Prove (a) $\Rightarrow$ (b), which is $b \rho c \Rightarrow [b] \cap [c] \neq \emptyset$

*Proof*

$$
\begin{array}{ll}
 & b \rho c \\
= & \langle \text{(3.39) Identity of } \wedge \rangle \\
 & true \wedge b \rho c \\
= & \langle \rho \text{ is reflexive} \rangle \\
 & b \rho b \wedge b \rho c \\
= & \langle \text{(14.34) Definition, twice} \rangle \\
 & b \in [b] \wedge b \in [c] \\
= & \langle \text{(11.21) Axiom intersection} \rangle \\
 & b \in [b] \cap [c] \\
\Rightarrow & \langle \text{Lemma: } b \in A \Rightarrow A \neq \emptyset \rangle \\
 & [b] \cap [c] \neq \emptyset \quad //
\end{array}
$$

Prove the lemma: $b \in A \Rightarrow A \neq \emptyset$

*Proof*

Use (4.12) Proof by contrapositive.

Must prove $A = \emptyset \Rightarrow \neg(b \in A)$

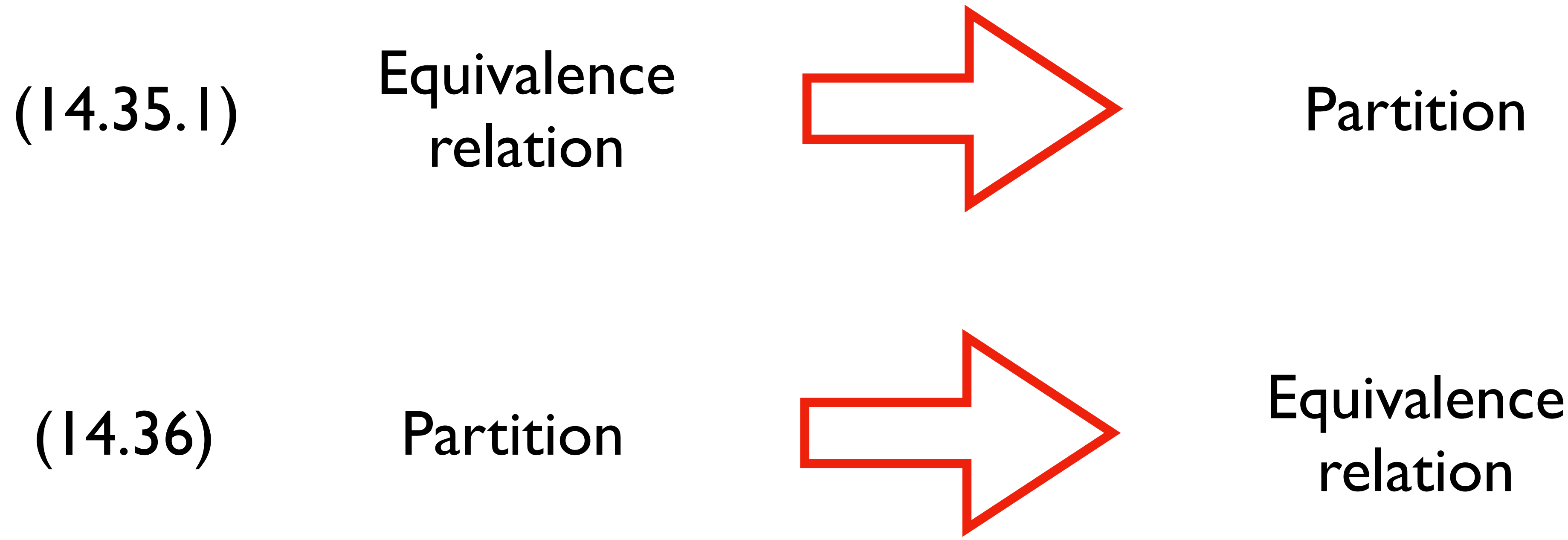
Use (4.4) Deduction. Assume the antecedent.

$$
\begin{array}{ll}
 & \neg(b \in A) \\
= & \langle \text{Assume antecedent } A = \emptyset \rangle \\
 & \neg(b \in \emptyset) \\
= & \langle (11.4.2) \rangle \\
 & \neg false \\
= & \langle (3.13) \rangle \\
 & true \quad //
\end{array}
$$

(14.35.1) Let $\rho$ be an equivalence relation on $B$. The equivalence classes partition $B$.

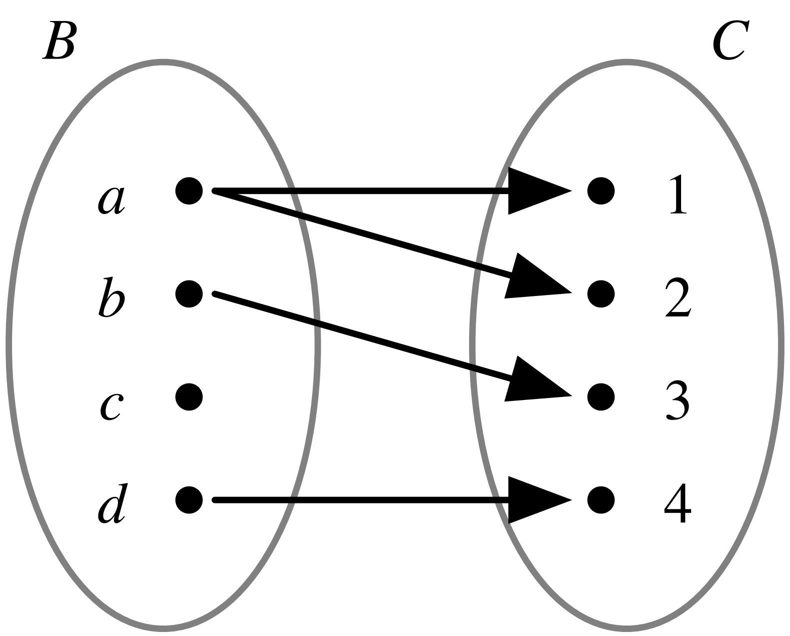
(14.36) Let $P$ be the set of sets of a partition of $B$. The following relation $\rho$ on $B$ is an equivalence relation:

$b\ \rho\ c \equiv (\exists p \mid p \in P : b \in p \wedge c \in p)$



(14.37) (a) **Definition:** A binary relation $f$ on $B \times C$ is *determinate* iff
$(\forall b, c, c' \mid b\ f\ c \wedge b\ f\ c' : c = c')$

(b) **Definition:** A binary relation is a *function* iff it is determinate.



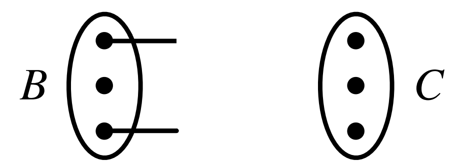

$\rho = \{\langle a,1\rangle, \langle a,2\rangle, \langle b,3\rangle, \langle d,4\rangle\}$

$\rho$ is a relation.

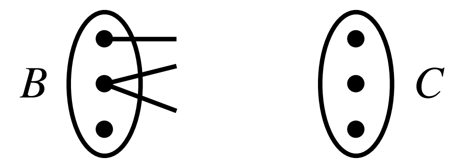
$\rho$ is not a function.

Have $a\rho 1 \wedge a\rho 2$ but $1 \neq 2$.

Determinate (14.37)




Determinate: $f$ is a function




Not determinate: $\rho$ is not a function

(14.37) (a) **Definition:** A binary relation $f$ on $B \times C$ is *determinate* iff
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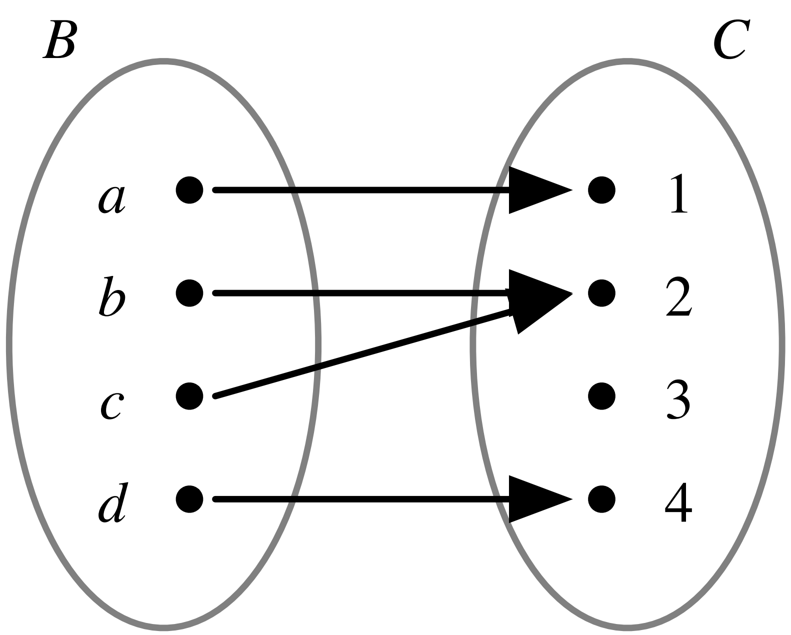


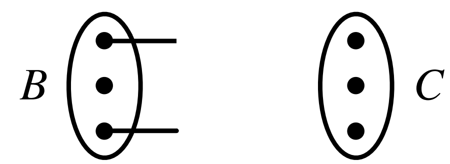

$f = \{\langle a,1\rangle, \langle b,2\rangle, \langle c,2\rangle, \langle d,4\rangle\}$

$f$ is a relation.

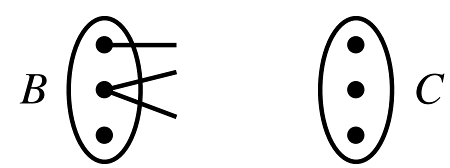
$f$ is a function.

$f : B \to C$

Determinate (14.37)

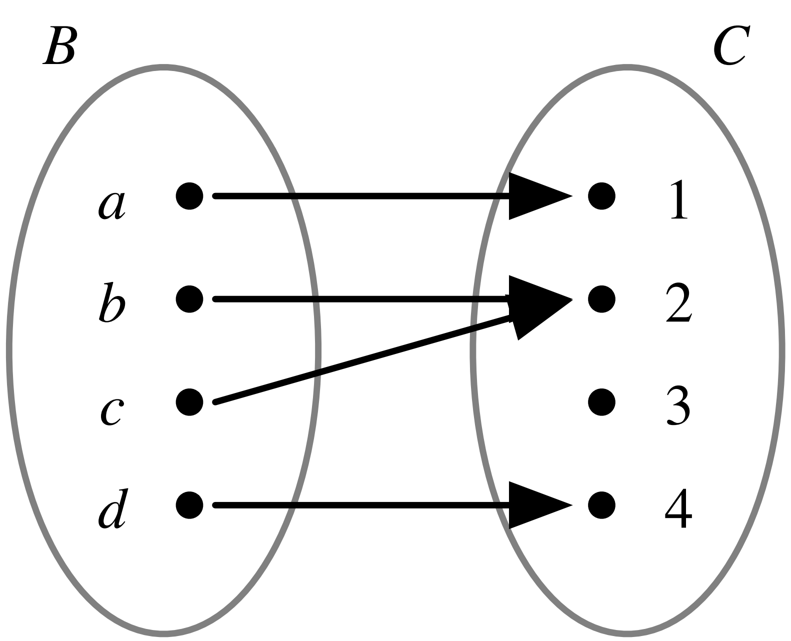



Determinate: $f$ is a function




Not determinate: $\rho$ is not a function

(14.37.1) **Notation:** $f.b = c$ and $b\ f\ c$ are interchangeable notations.



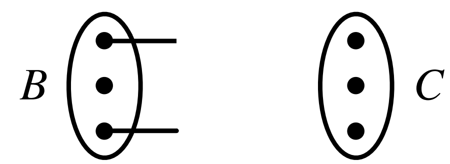
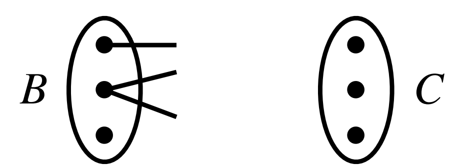

$f = \{\langle a,1\rangle, \langle b,2\rangle, \langle c,2\rangle, \langle d,4\rangle\}$

$f$ is a relation.

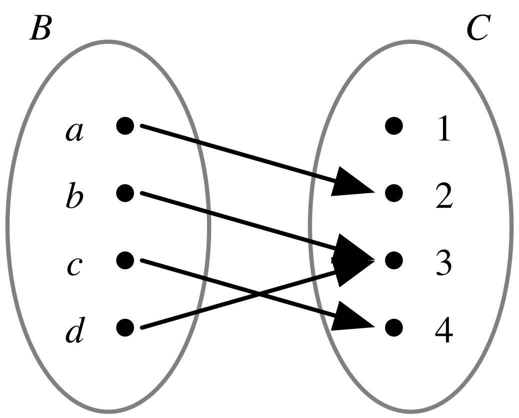
$f$ is a function.

$f : B \to C$

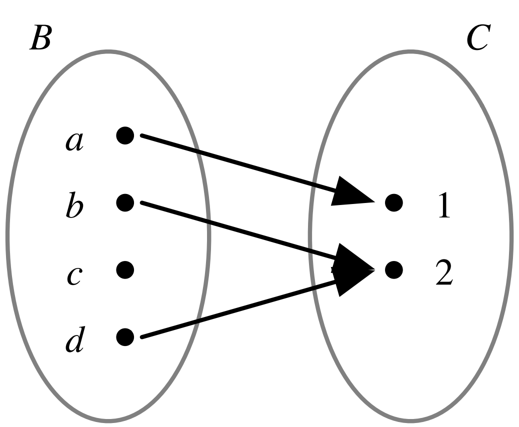
$f.d = 4$ is equivalent to $d\ f\ 4$

| | |
|---|---|
| Determinate (14.37)   Determinate: $f$ is a function   Not determinate: $\rho$ is not a function | |
| | |

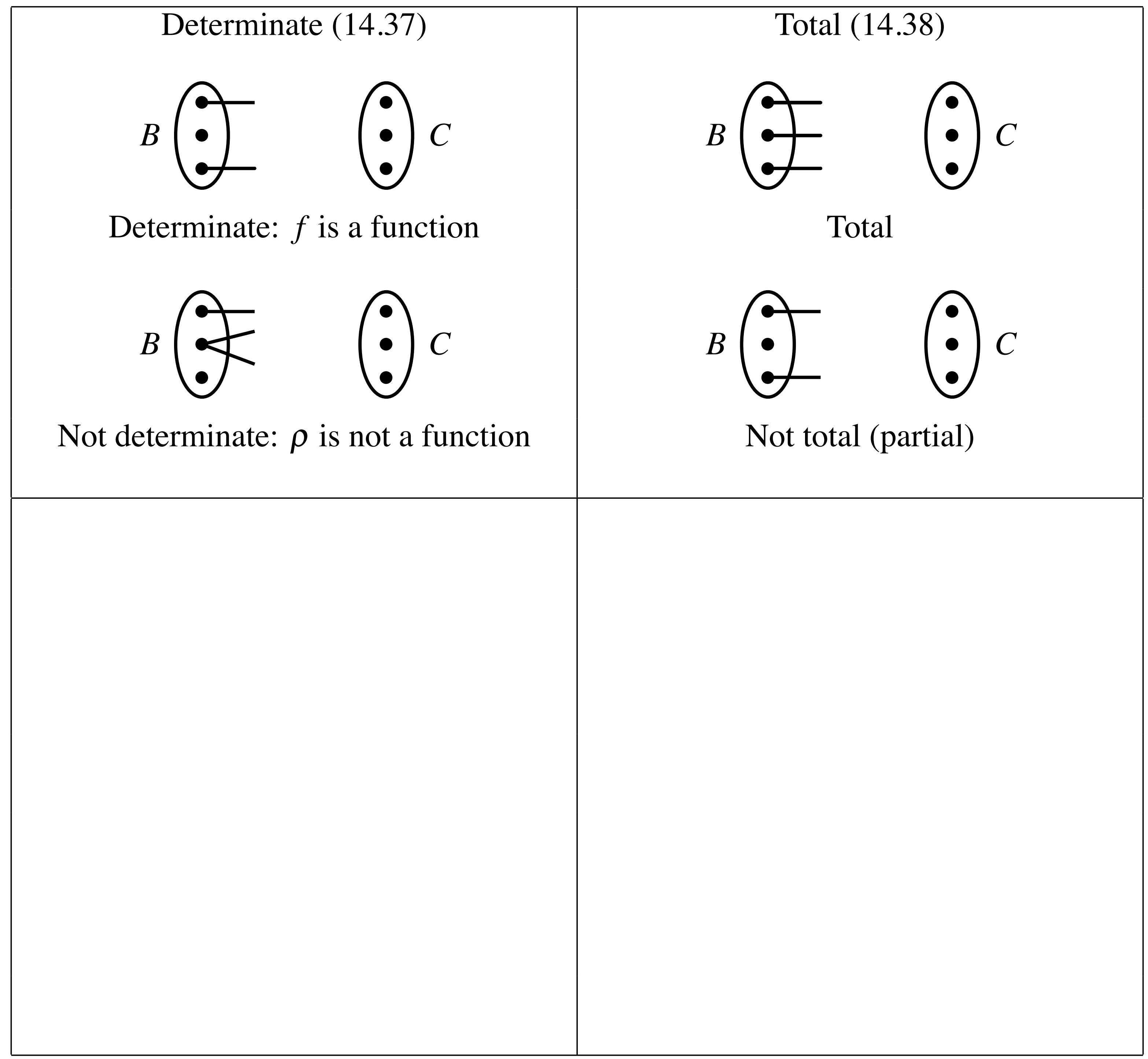
(14.38) **Definition:** A function $f$ on $B \times C$ is *total* if $B = Dom.f$.
Otherwise it is *partial*.
We write $f : B \to C$ for the type of $f$ if $f$ is total and $f : B \rightsquigarrow C$ if $f$ is partial.




$f$ is total.
$f : B \to C$



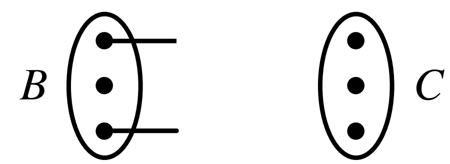
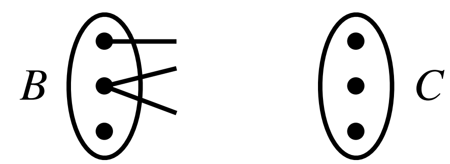
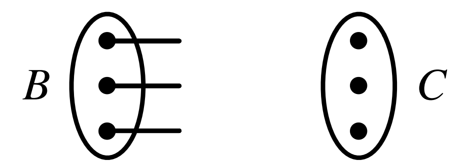
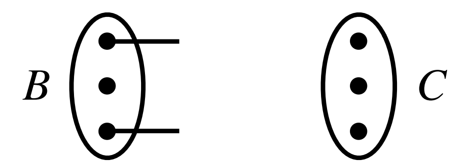

$f$ is partial.
$f : B \rightsquigarrow C$



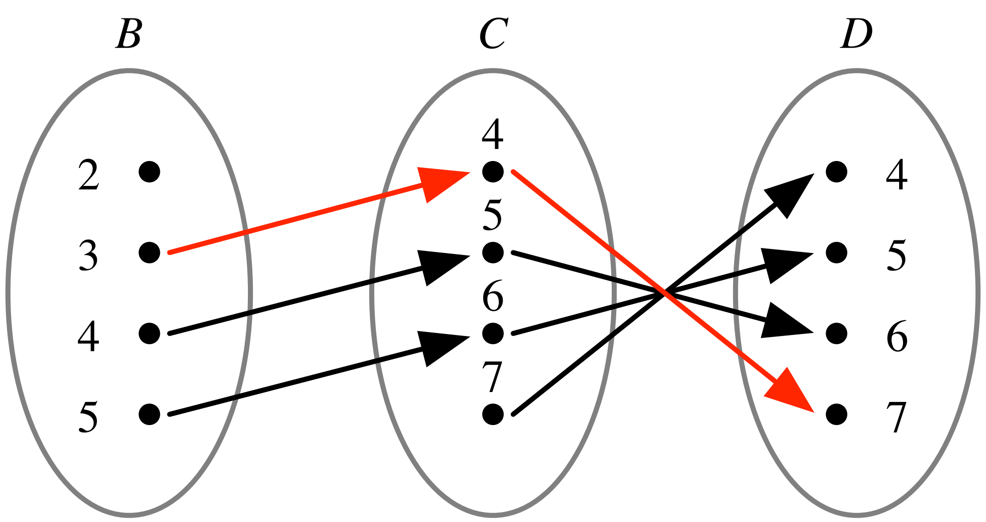
# A Logical Approach to Discrete Math

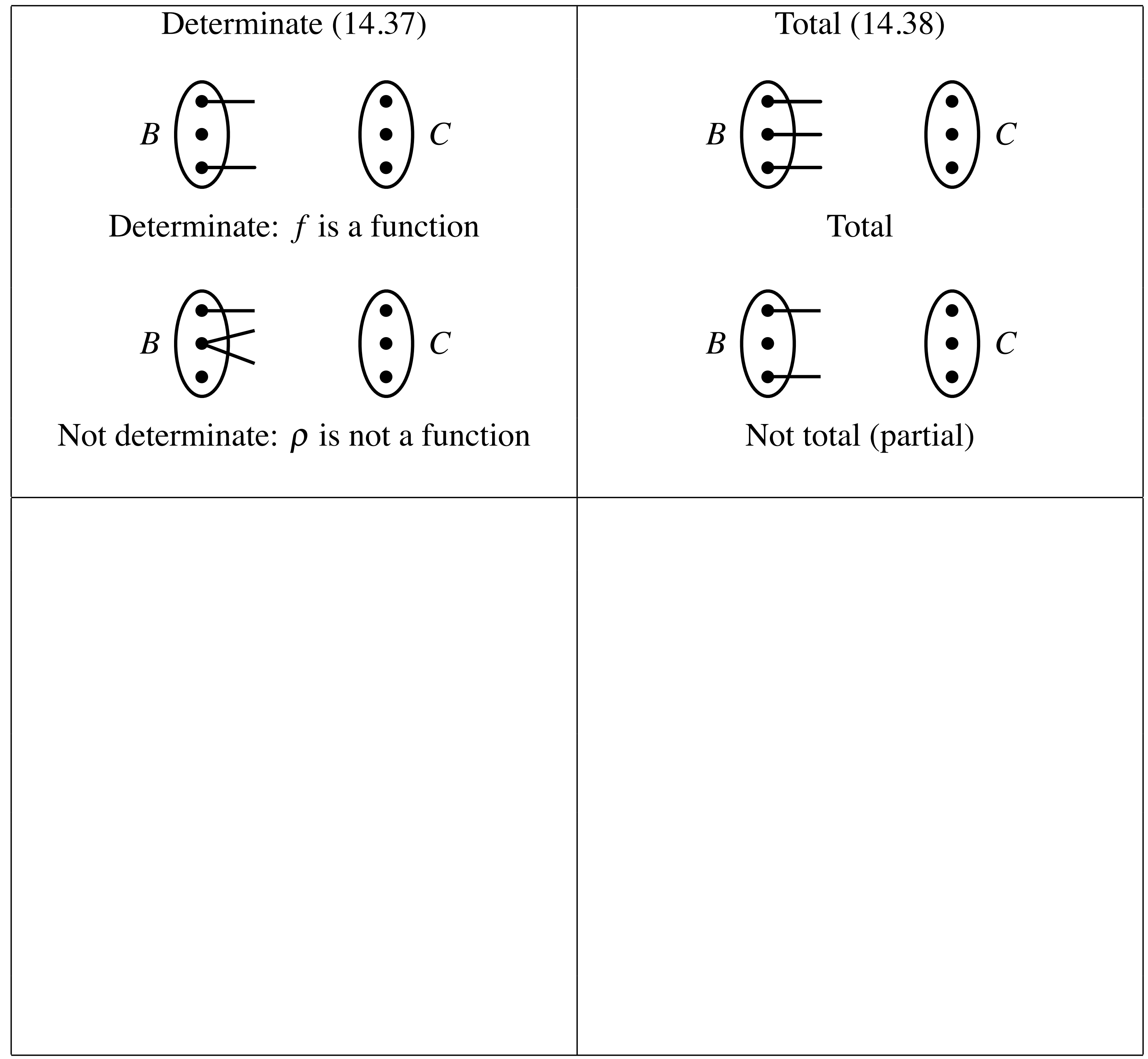
(14.38.1) **Total:** A function $f$ on $B \times C$ is total if, for an arbitrary element $b$: $B$,
$(\exists c \colon C \mid \colon f.b = c)$

Homework

| Determinate (14.37) | Total (14.38) |
|---|---|
| <br>Determinate: $f$ is a function<br><br>Not determinate: $\rho$ is not a function | <br>Total<br><br>Not total (partial) |
| | |

(14.39) **Definition, Composition:** For functions $f$ and $g$, $f \bullet g = g \circ f$.



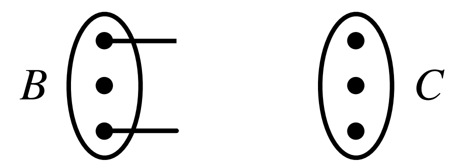
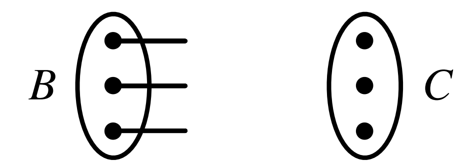
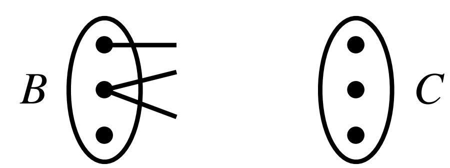
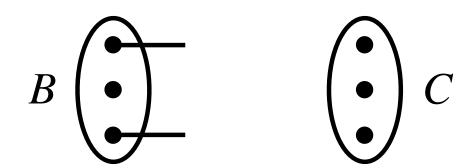

$3\ (pred \circ swap)\ 7$

$3\ (swap \bullet pred)\ 7$ by (14.39)

$(swap \bullet pred).3 = 7$ by (14.37.1)

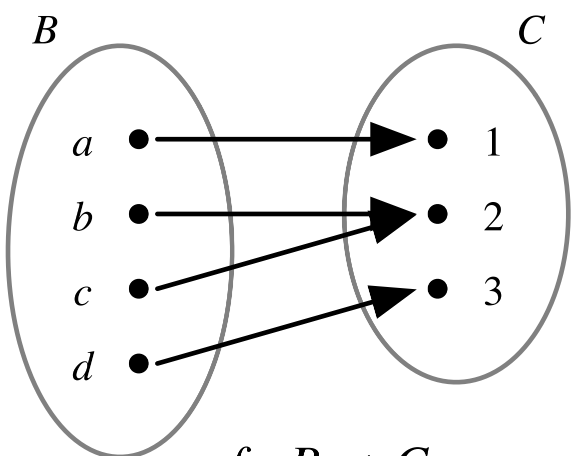
(14.40) Let $g : B \to C$ and $f : C \to D$ be total functions.
Then the composition $f \bullet g$ of $f$ and $g$ is the total function defined by
$(f \bullet g).b = f(g.b)$

Homework

| Determinate (14.37) | Total (14.38) |
|---|---|
|   |   |
| Determinate: $f$ is a function | Total |
|   |   |
| Not determinate: $\rho$ is not a function | Not total (partial) |

(14.41) **Definitions:** (a) Total function $f : B \to C$ is *onto* or *surjective* if $Ran.f = C$.

(b) Total function $f$ is *one-to-one* or *injective* if $(\forall b, b' \colon B, c \colon C \mid \colon b \; f \; c \land b' \; f \; c \equiv b = b')$.

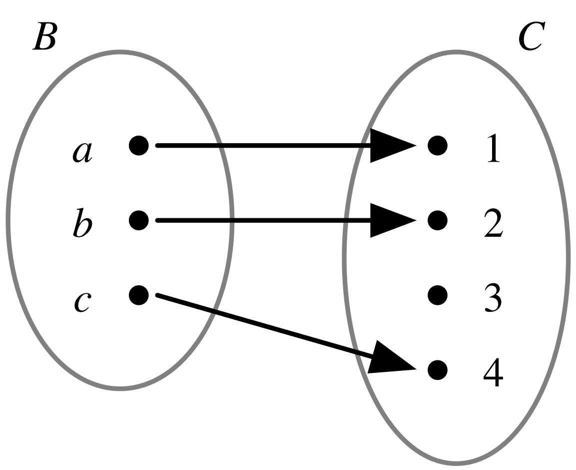



$f : B \to C$

$f$ is total.

$f$ is onto.

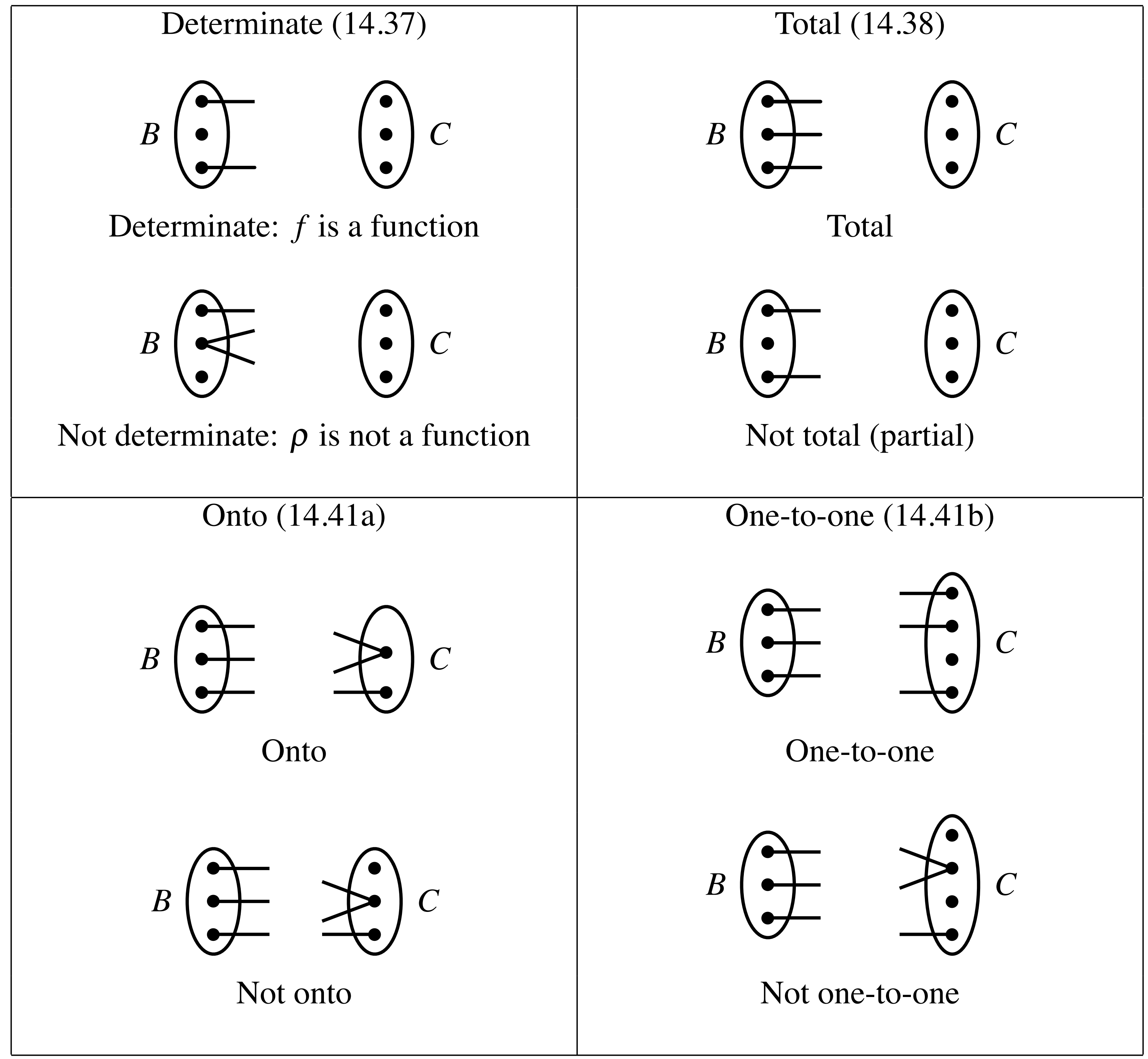
$f$ is not one-to-one.




$f : B \to C$

$f$ is total.

$f$ is not onto.



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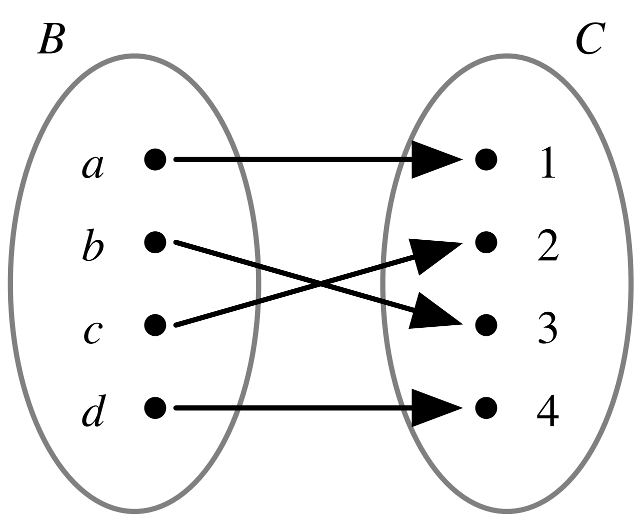
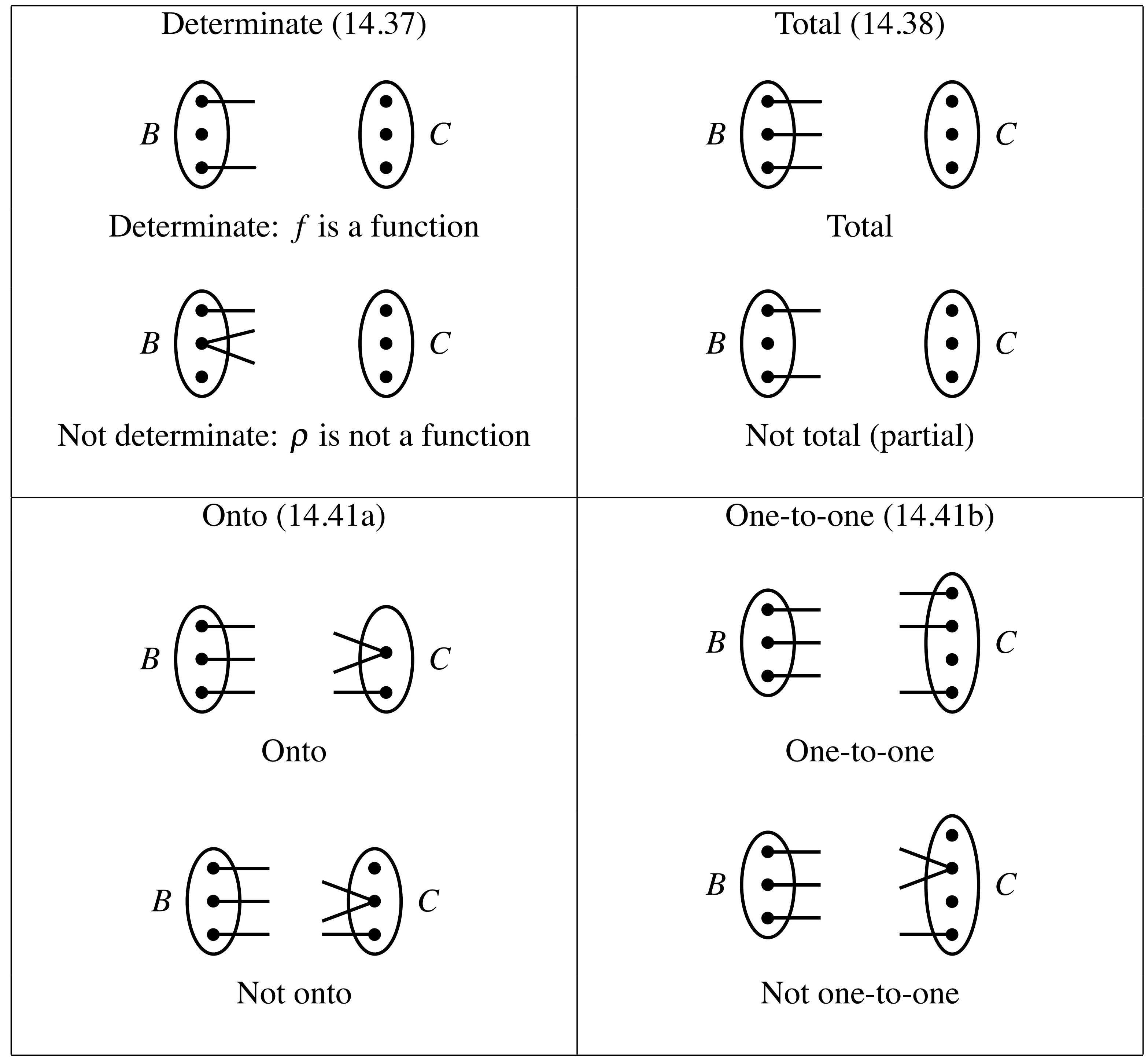



$f : B \to C$

$f$ is total.

$f$ is onto.

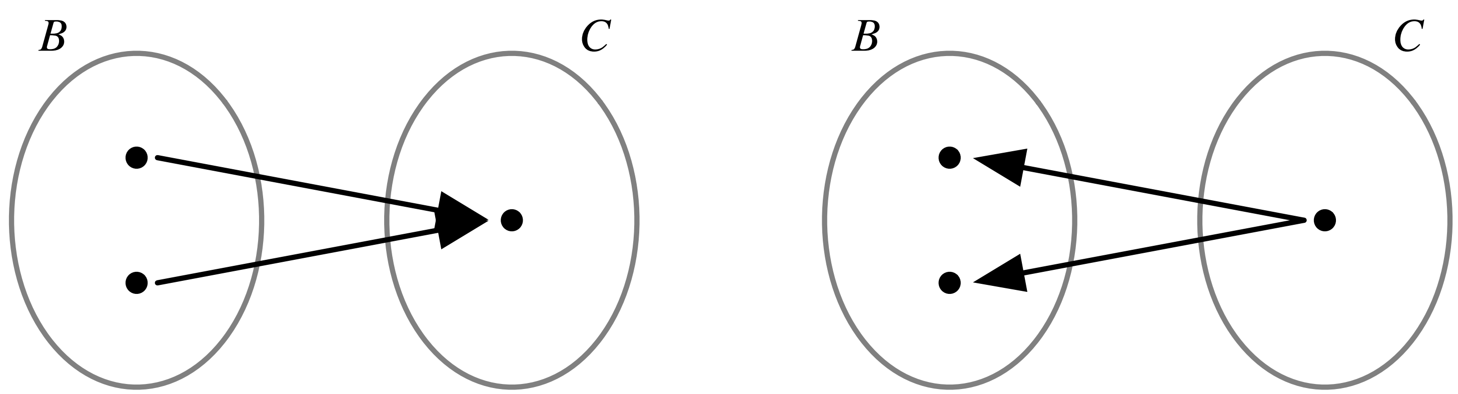
$f$ is one-to-one.



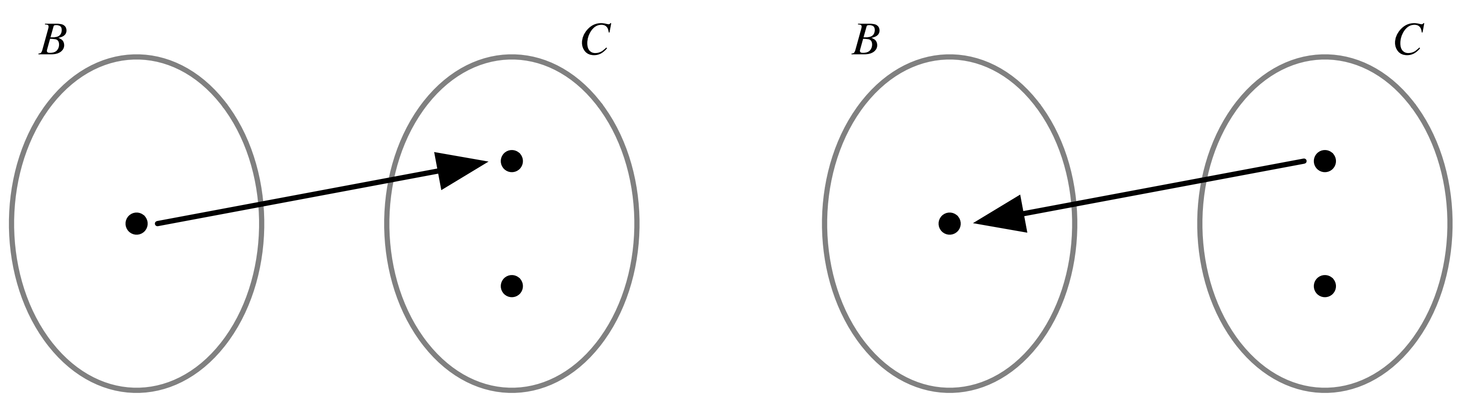
(14.42) Let $f$ be a total function, and let $f^{-1}$ be its relational inverse.

(a) Then $f^{-1}$ is a function, i.e. is determinate, iff $f$ is one-to-one.

(b) And, $f^{-1}$ is total iff $f$ is onto.




(a) $f$ is not one-to-one. $f^{-1}$ is not determinate.



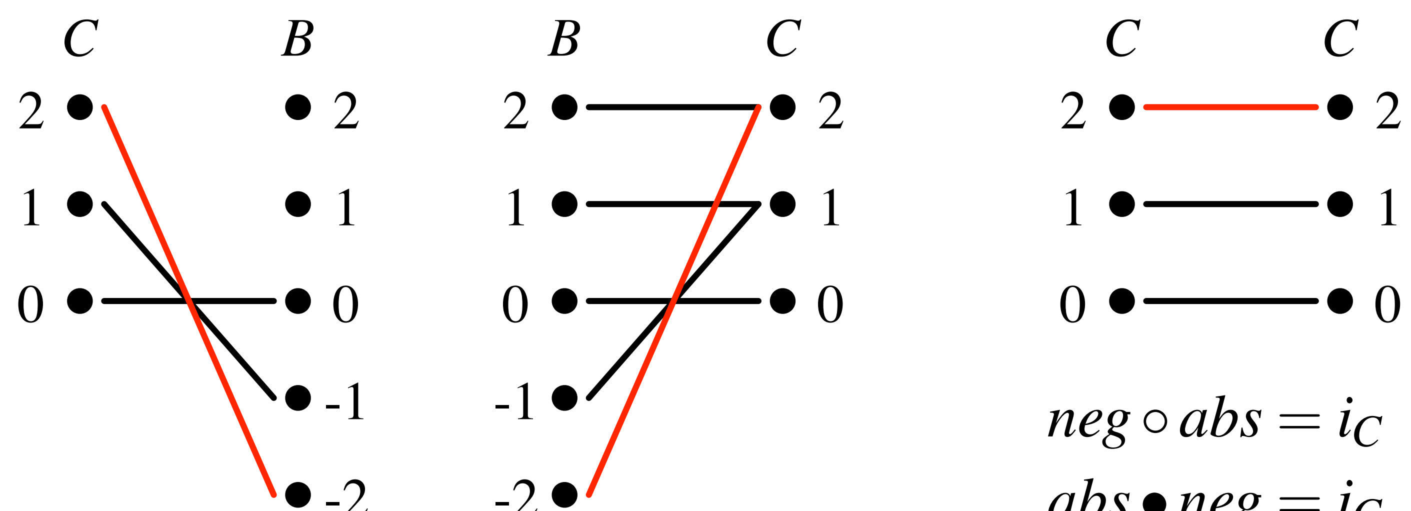

(b) $f$ is not onto. $f^{-1}$ is not total.

(14.43) **Definitions:** Let $f : B \to C$.

(a) A *left inverse* of $f$ is a function $g : C \to B$ such that $g \bullet f = i_B$.

(b) A *right inverse* of $f$ is a function $g : C \to B$ such that $f \bullet g = i_C$.

(c) Function $g$ is an *inverse* of $f$ if it is both a left inverse and a right inverse.




$neg : C \to B$

$abs : B \to C$

$neg \circ abs = i_C$

$abs \bullet neg = i_C$

*neg* is a right inverse of *abs*.

(14.47) **Definition:** A binary relation $\rho$ on a set $B$ is called a *partial order on b* if it is reflexive, antisymmetric, and transitive. In this case, pair $\langle B, \rho \rangle$ is called a *partially ordered set* or *poset*.

We use the symbol $\preceq$ for an arbitrary partial order, sometimes writing $c \succeq b$ instead of $b \preceq c$.

Equivalence relation:
- Reflexive
- Symmetric
- Transitive

Partial order:
- Reflexive
- Antisymmetric
- Transitive

(14.47) **Definition:** A binary relation $\rho$ on a set $B$ is called a *partial order on b* if it is reflexive, antisymmetric, and transitive. In this case, pair $\langle B, \rho \rangle$ is called a *partially ordered set* or *poset*.

We use the symbol $\preceq$ for an arbitrary partial order, sometimes writing $c \succeq b$ instead of $b \preceq c$.

Example 1

$B: \{a, b, c\}$

$\mathcal{P}B = \{\{\}, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$

$\langle \mathcal{P}B, \subseteq \rangle$ is a poset.

Reflexive: $D \subseteq D$

Antisymmetric: $D \subseteq E \wedge E \subseteq D \Rightarrow D = E$

Transitive: $D \subseteq E \wedge E \subseteq F \Rightarrow D \subseteq F$

(14.47) **Definition:** A binary relation $\rho$ on a set $B$ is called a *partial order on b* if it is reflexive, antisymmetric, and transitive. In this case, pair $\langle B, \rho \rangle$ is called a *partially ordered set* or *poset*.

We use the symbol $\preceq$ for an arbitrary partial order, sometimes writing $c \succeq b$ instead of $b \preceq c$.

<u>Example 2</u>

$B: \{3, 4, 6, 8, 12, 24\}$

$\langle B, | \rangle$ where $|$ means "divides" is a poset.

Reflexive: $b \mid b$

Antisymmetric: $b \mid c \wedge c \mid b \Rightarrow b = c$

Transitive: $b \mid c \wedge c \mid d \Rightarrow b \mid d$

(14.47) **Definition:** A binary relation $\rho$ on a set $B$ is called a *partial order on b* if it is reflexive, antisymmetric, and transitive. In this case, pair $\langle B, \rho \rangle$ is called a *partially ordered set* or *poset*.

We use the symbol $\preceq$ for an arbitrary partial order, sometimes writing $c \succeq b$ instead of $b \preceq c$.

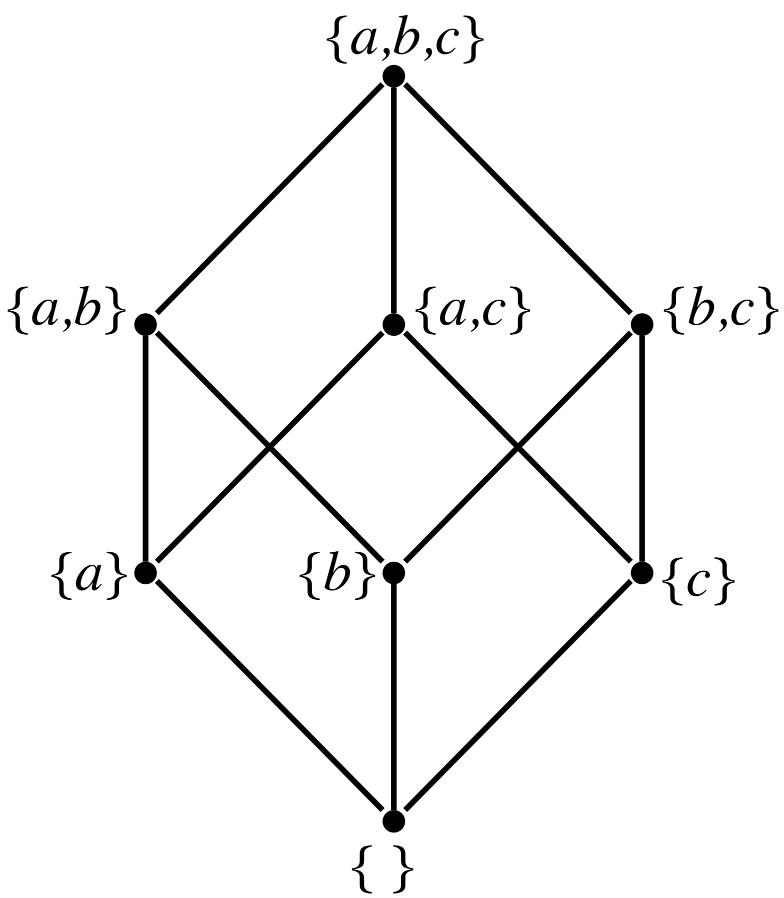
Hasse diagrams

- Each element in $B$ is a dot.
- Elevation matters.
- If $b \preceq c$ there is a line up from $b$ to $c$,
  but only if there is not another element $d$
  that is "between" $b$ and $c$ such that $b \preceq d \preceq c$.

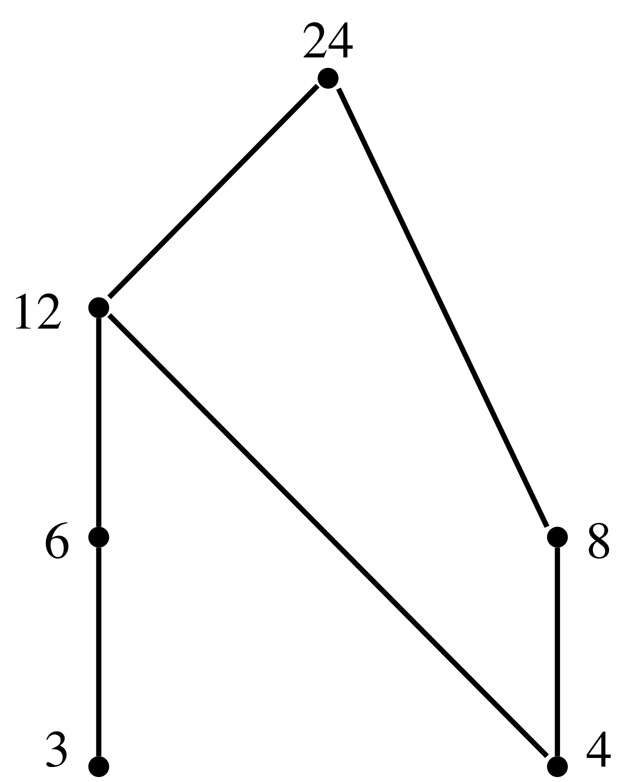
(14.47) **Definition:** A binary relation $\rho$ on a set $B$ is called a *partial order on b* if it is reflexive, antisymmetric, and transitive. In this case, pair $\langle B, \rho \rangle$ is called a *partially ordered set* or *poset*.

We use the symbol $\preceq$ for an arbitrary partial order, sometimes writing $c \succeq b$ instead of $b \preceq c$.

Example 1




Example 2



(14.47.1) **Definition, Incomparable:** $\text{incomp}(b,c) \equiv \neg(b \preceq c) \wedge \neg(c \preceq b)$

<u>Example 1</u>

$\{a,b\}$ and $\{a,c\}$ are incomparable.

$\neg(\{a,b\} \subseteq \{a,c\}) \wedge \neg(\{a,c\} \subseteq \{a,b\})$

<u>Example 2</u>

6 and 8 are incomparable.

$\neg(6 \mid 8) \wedge \neg(8 \mid 6)$

(14.48) **Definition:** Relation $\prec$ is a *quasi order* or *strict partial order* if $\prec$ is transitive and irreflexive

Example

The proper subset relation $\subset$ is a quasi order.

Irreflexive: $\neg(D \subset D)$

Transitive: $D \subset E \wedge E \subset F \Rightarrow D \subset F$

(14.48) **Definition:** Relation $\prec$ is a *quasi order* or *strict partial order* if $\prec$ is transitive and irreflexive

(14.47) $\preceq$ is a RAT relation (partial order).

(14.48) $\prec$ is a IT relation (strict partial order).

We can prove that a IT relation is also antisymmetric.

Therefore, $\prec$ is a IAT relation.

Summary

Equivalence relation $=$ is RST.

Partial order $\preceq$ is RAT.

Strict partial order $\prec$ is IAT.

(14.48.1) **Definition, Reflexive reduction:** Given $\preceq$, its *reflexive reduction* $\prec$ is computed by eliminating all pairs $\langle b,b\rangle$ from $\preceq$.

(14.48.2) Let $\prec$ be the reflexive reduction of $\preceq$. Then,
$\neg(b \preceq c) \equiv c \prec b \vee \text{incomp}(b,c)$

(14.49) (a) If $\rho$ is a partial order over a set $B$, then $\rho - i_B$ is a quasi order.
(b) If $\rho$ is a quasi order over a set $B$, then $\rho \cup i_B$ is a partial order.

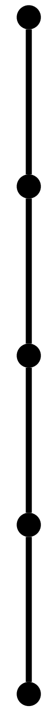
Reflexive reduction is the opposite of reflexive closure.

To compute the reflexive closure of a relation, you add ordered pairs to make the relation reflexive.

To compute the reflexive reduction of a relation, you eliminate ordered pairs to make the relation irreflexive.

(14.50) **Definition:** A partial order $\preceq$ over $B$ is called a *total* or *linear* order if $(\forall b, c \,|\!: b \preceq c \vee b \succeq c)$, i.e. iff $\preceq \cup \preceq^{-1} = B \times B$.
In this case, the pair $\langle B, \preceq \rangle$ is called a *linearly ordered set* or a *chain*.

Hasse diagram of a total order.
All pairs of elements are comparable.



Examples

$\langle \mathbb{N}, \leq \rangle$ is a total order.
$\langle \{1,3,6,9,12\}, | \rangle$ is not a total order.
$\langle \{1,3,6,12,24\}, | \rangle$ is a total order.

(14.51) **Definitions:** Let $S$ be a nonempty subset of poset $\langle U, \preceq \rangle$.

(a) Element $b$ of $S$ is a *minimal element of* $S$ if no element of $S$ is smaller than $b$, i.e. if $b \in S \land (\forall c \mid c \prec b : c \notin S)$.

(b) Element $b$ of $S$ is the *least element of* $S$ if $b \in S \land (\forall c \mid c \in S : b \preceq c)$.

(c) Element $b$ is a *lower bound of* $S$ if $(\forall c \mid c \in S : b \preceq c)$. (A lower bound of $S$ need not be in $S$.)

(d) Element $b$ is the *greatest lower bound of* $S$, written $glb.S$ if $b$ is a lower bound and if every lower bound $c$ satisfies $c \preceq b$.

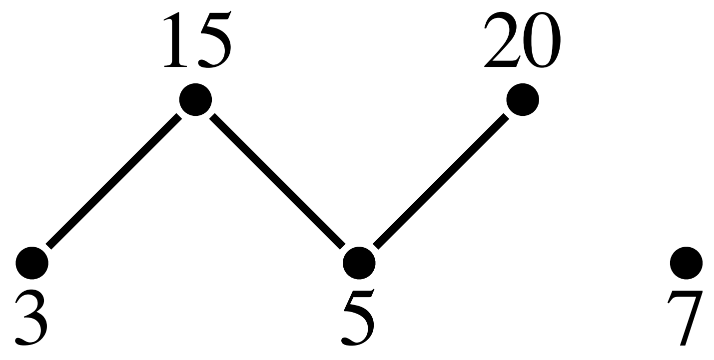
Example

In $\langle \mathbb{N}, | \rangle$ with $S = \{3, 5, 7, 15, 20\}$

$3, 5, 7$ are minimal.

There is no least element.

For $b$ to be least it must be related to every other element.



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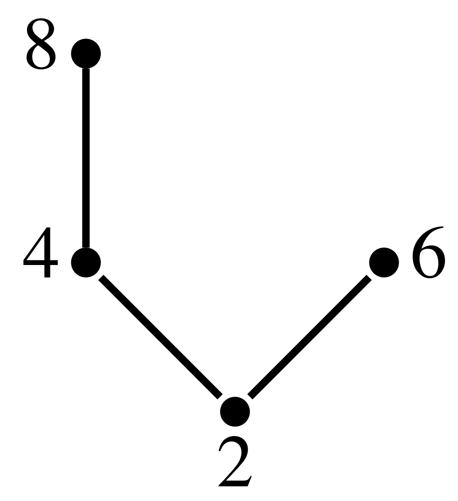
(d) Element $b$ is the *greatest lower bound of $S$*, written $glb.S$ if $b$ is a lower bound and if every lower bound $c$ satisfies $c \preceq b$.

Example

In $\langle \mathbb{N}, | \rangle$ with $S = \{2, 4, 6, 8\}$

2 is minimal and least.



# A Logical Approach to Discrete Math

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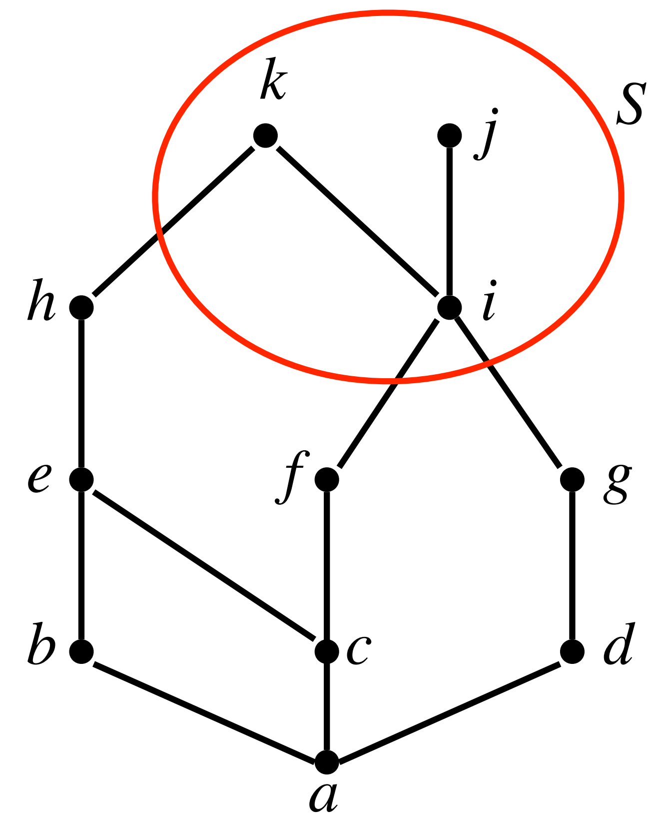
(d) Element $b$ is the *greatest lower bound of* $S$, written $glb.S$ if $b$ is a lower bound and if every lower bound $c$ satisfies $c \preceq b$.

<u>Example</u>

In set $B = \{a,b,c,d,e,f,g,h,i,j,k\}$ with the relation defined by the Hasse diagram and subset $S = \{i,j,k\}$ the lower bounds of $\{i,j,k\}$ are $i,f,g,c,d,a$. The greatest lower bound is $glb.S = i$.



(14.51) **Definitions:** Let $S$ be a nonempty subset of poset $\langle U, \preceq \rangle$.

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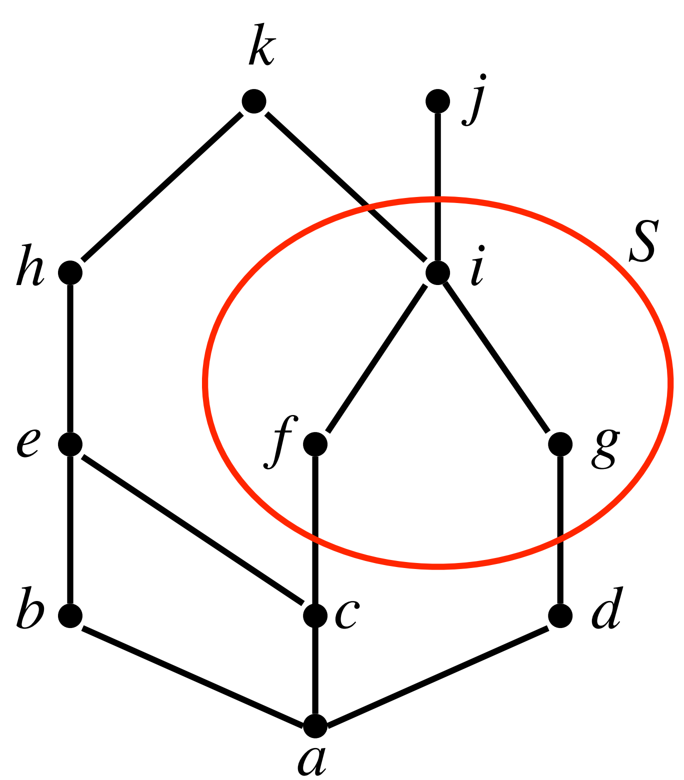
<u>Example</u>

In set $B = \{a,b,c,d,e,f,g,h,i,j,k\}$ with the relation defined by the Hasse diagram and subset $S = \{i,j,k\}$ the lower bounds of $\{i,j,k\}$ are $i,f,g,c,d,a$.
The greatest lower bound is $glb.S = i$.
The lower bound of $S = \{i,f,g\}$ is only $a$.



(14.52) Every finite nonempty subset $S$ of poset $\langle U, \preceq \rangle$ has a minimal element.

(14.53) Let $B$ be a nonempty subset of poset $\langle U, \preceq \rangle$.

(a) A least element of $B$ is also a minimal element of $B$ (but not necessarily vice versa). Homework

(b) A least element of $B$ is also a greatest lower bound of $B$ (but not necessarily vice versa).

(c) A lower bound of $B$ that belongs to $B$ is also a least element of $B$.

((14.54) **Definitions:** Let $S$ be a nonempty subset of poset $\langle U, \preceq \rangle$.

(a) Element $b$ of $S$ is a *maximal element of* $S$ if no element of $S$ is larger than $b$, i.e. if $b \in S \wedge (\forall c \mid b \prec c : c \notin S)$.

(b) Element $b$ of $S$ is the *greatest element of* $S$ if $b \in S \wedge (\forall c \mid c \in S : c \preceq b)$.

(c) Element $b$ is an *upper bound of* $S$ if $(\forall c \mid c \in S : c \preceq b)$.
(An upper bound of $S$ need not be in $S$.)

(d) Element $b$ is the *least upper bound of* $S$, written $lub.S$, if $b$ is an upper bound and if every upper bound $c$ satisfies $b \preceq c$.

# Relational databases

Binary relation

Subset of ordered pairs from $B_1 \times B_2$

Trinary relation

Subset of ordered triples from $B_1 \times B_2 \times B_3$

*n*-ary relation

Subset of ordered $n$-tuples from $B_1 \times B_2 \times B_3 \ldots \times B_n$

# Relational database tables

Relation

MyRelation $= \{\langle$apple, baseball, cat, John $\rangle, \langle$banana, football, dog, Mary $\rangle\}$

Table representation

MyRelation

| | | | |
|---|---|---|---|
| apple | baseball | cat | John |
| banana | football | dog | Mary |

Table representation with field names

MyRelation

| Fruit | Toy | Animal | Person |
|---|---|---|---|
| apple | baseball | cat | John |
| banana | football | dog | Mary |

LADM has three relational database examples in Chapter 14. Each database has a group of relations, represented by tables, and each relation has a name. Below are the first two *n*-tuples in each relation in each database.

Example A. Two tables: PABM and MC.

PABM

| Title | Month | Day | Year | Theater | Perfs |
|---|---|---|---|---|---|
| My Fair Lady | 3 | 15 | 1956 | Mark Hellinger | 2717 |
| Man of La Mancha | 11 | 22 | 1965 | ANTA Wash. Sq. | 2329 |

MC

| Title | Book | Lyrics | Music |
|---|---|---|---|
| My Fair Lady | Lerner | Lerner | Loewe |
| Man of La Mancha | Wasserman | Darion | Leigh |

# A Logical Approach to Discrete Math

TABLE 14.2. Popular American Broadway Musicals (*PABM*)

| Title | Opening Month | Day | Year | Theater | Perfs |
|---|---|---|---|---|---|
| My Fair Lady | 3 | 15 | 1956 | Mark Hellinger | 2717 |
| Man of La Mancha | 11 | 22 | 1965 | ANTA Wash. Sq. | 2329 |
| Oklahoma! | 3 | 31 | 1943 | St. James | 2248 |
| Hair | 4 | 29 | 1968 | Biltmore | 1750 |
| The King and I | 3 | 29 | 1951 | St. James | 1246 |
| Guys and Dolls | 11 | 24 | 1950 | Forty-Sixth St. | 1200 |
| Cabaret | 11 | 20 | 1966 | Broadhurst | 1166 |
| Damn Yankees | 5 | 5 | 1955 | Forty-Sixth St. | 1019 |
| Camelot | 12 | 3 | 1960 | Majestic | 878 |
| West Side Story | 9 | 26 | 1957 | Winter Garden | 732 |

TABLE 14.3. Musical Creators (*MC*)

| Title | Book | Lyrics | Music |
|---|---|---|---|
| My Fair Lady | Lerner | Lerner | Loewe |
| Man of La Mancha | Wasserman | Darion | Leigh |
| Oklahoma! | Hammerstein | Hammerstein | Rodgers |
| Hair | Ragni & Rado | Ragni & Rado | MacDermot |
| The King and I | Hammerstein | Hammerstein | Rodgers |
| Guys and Dolls | Swerling & Burrows | Loesser | Loesser |
| Cabaret | Masteroff | Ebb | Kander |
| Damn Yankees | Abbott & Wallop | Adler & Ross | Adler & Ross |
| Camelot | Lerner | Lerner | Loewe |
| West Side Story | Laurents | Sondheim | Bernstein |

# A Logical Approach to Discrete Math

$$PABM = Title \times Month \times Day \times Year \times Theater \times Perfs$$

*Title* is the set of titles for Broadway shows;
*Month* is the set 1..12 corresponding to the months of the year;
*Day* is the set 1..31 corresponding to the days of the months;
*Year* is the set $\mathbb{Z}^+$ of positive integers;
*Theater* is the set of theaters in and around Broadway, NYC;
*Perfs* is the set $\mathbb{Z}^+$ of positive integers.

$PABM(Title, Month, Day, Year, Theater, Perfs)$

$MC(Title, Book, Lyrics, Music)$

# A Logical Approach to Discrete Math

Example B. One table: ALL.

ALL

| Title | Month | Day | Year | Theater | Perfs | Book | Lyrics | Music |
|---|---|---|---|---|---|---|---|---|
| My Fair Lady | 3 | 15 | 1956 | Mark Hellinger | 2717 | Lerner | Lerner | Loewe |
| Man of La Mancha | 11 | 22 | 1965 | ANTA Wash. Sq. | 2329 | Wasserman | Darion | Leigh |

$$ALL(Title, Month, Day, Year, Theater, Perfs, Book, Lyrics, Music) \quad .$$

# A Logical Approach to Discrete Math

Example C. Six tables: Where, When, Author, Run, Lyricist, and Composer.

Where

| Title | Theater |
|---|---|
| My Fair Lady | Mark Hellinger |
| Man of La Mancha | ANTA Wash. Sq. |

When

| Title | Month | Day | Year |
|---|---|---|---|
| My Fair Lady | 3 | 15 | 1956 |
| Man of La Mancha | 11 | 22 | 1965 |

Author

| Title | Book |
|---|---|
| My Fair Lady | Lerner |
| Man of La Mancha | Wasserman |

Run

| Title | Perfs |
|---|---|
| My Fair Lady | 2717 |
| Man of La Mancha | 2329 |

Lyricist

| Title | Lyrics |
|---|---|
| My Fair Lady | Lerner |
| Man of La Mancha | Darion |

Composer

| Title | Music |
|---|---|
| My Fair Lady | Loewe |
| Man of La Mancha | Leigh |

# A Logical Approach to Discrete Math

$$\begin{array}{l} Where(Title, Theater) \\ When(Title, Month, Day, Year) \\ Author(Title, Book) \\ Run(Title, Perfs) \\ Lyricist(Title, Lyrics) \\ Composer(Title, Music) \quad . \end{array}$$

(14.56.1) **Definition, select:** For Relation $R$ and predicate $F$, which may contain names of fields of $R$, $\sigma(R, F) = \{t \mid t \in R \wedge F\}$

(14.56.2) **Definition, project:** For $A_1, \ldots, A_m$ a subset of the names of the fields of relation $R$, $\pi(R, A_1, \ldots, A_m) = \{t \mid t \in R : \langle t.A_1, t.A_2, \ldots, t.A_m \rangle\}$

(14.56.3) **Definition, natural join:** For Relations $R1$ and $R2$, $R1 \bowtie R2$ has all the attributes that $R1$ and $R2$ have, but if an attribute appears in both, then it appears only once in the result; further, only those tuples that agree on this common attribute are included.

Select

$\sigma$ selects rows from $R$ that satisfy $F$.

Example: Use database A to list all the 6-tuples that opened on Forty-Sixth St.

$\sigma$(PABM, Theater = Forty-Sixth St.)

Project

$\pi$ selects fields (attributes) from $R$ as listed.

Example: Use database A to list only the titles of the musicals that opened on Forty-Sixth St.

$\pi$($\sigma$(PABM, Theater = Forty-Sixth St.), Title)

(14.56.1) **Definition, select:** For Relation $R$ and predicate $F$, which may contain names of fields of $R$, $\sigma(R,F) = \{t \mid t \in R \wedge F\}$

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Join

$\bowtie$ is a binary infix operator.

Example: Use database C to list the theater where each book was performed.

Author $\bowtie$ Where has three columns: Title, Book, Theater.

To list just the Book and Theater

$\pi$(Author $\bowtie$ Where, Book, Theater)

Example: Use database A to list who wrote the lyrics for the show that had 2717 performances.

$\pi(\sigma$(PABM $\bowtie$ MC, Perfs = 2717), Lyrics)